FTD-HT-23-51-68

AD683365

FOREIGN TECHNOLOGY DIVISION

FUNDAMENTALS OF OPERATIONS RESEARCH IN COMBAT MATERIEL AND WEAPONRY

by

Yu. V. Chuyev, P. M. Mel'nikov, S. I. Petukhov, G. F. Stepanov and Ya. B. Shor

(VOL. ONE OF TWO VOLUMES)

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EDITED TRANSLATION

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Translated under: Contract F33(657)-68-D-0865

VOL. ONE OF TWO VOLUMES

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

FTD- HT -23-51-68

Date 10 Dec 19 68

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OSNOVY ISSLEDOVANIYA OPERATSIY V VOYENNOY TEKHNIKE

Pod Obshchey Redaktsiyei

Doktora Tekhnicheskikh Nauk, Prof.

Yu. V. Chueva

Izdatel'stvo "Sovetskoye Radio"

Moskva - 1965

591 pages

DATA HANDLING PAGE

01-ACCESSION NO.	98-DOCUMENT LOC	39-TOPIC TAGS
TM8001389		military engineering, military operation, military R and D

09-TITLE FUNDAMENTALS OF OPERATIONS RESEARCH IN COMBAT MATERIEL AND WEAPONRY

47-SUBJECT AREA 19

42-AUTHOR/CO-AUTHORS	10-DATE OF INFO
CHUYEV, YU. V.; 16-MEL'NIKOV, P. M.; 16-PETUKHOV, S.I.; 16-STEPANOV, G. F.; 16-SHOR, YA. B.	-----65

43-SOURCE	68-DOCUMENT NO.	69-PROJECT NO.
OSNOVY ISSLEDOVANIYA OPERATSIY V VOYENNOY TEKHNIKE. MOSCOW, IZD-VO SOVETSKOYE RADIO (RUSSIAN)	FTD-HT-23-51-68	31200

63-SECURITY AND DOWNGRADING INFORMATION	64-CONTROL MARKINGS	97-HEADER CLASN
UNCL, 0	NONE	UNCL

76-REEL/FRAME NO.	77-SUPERSEDES	78-CHANGES	40-GEOGRAPHICAL AREA	NO. OF PAGES
1886 0442			UR	481

CONTRACT NO.	X REF ACC. NO.	PUBLISHING DATE	TYPE PRODUCT	REVISION FREQ
F33657-68-D-0865	65-BX7014159	94-00	Translation	NONE

STEP NO.	ACCESSION NO.
02-UR/0000/65/000/000/0001/0591	

ABSTRACT The book reviews the fundamental characteristics of military engineering units, gives certain methods for evaluating their reliability and effectiveness in various combat and operational situations by using various analytical methods and statistical modelling methods with electronic computers.

The authors consider the tasks of evaluating the effectiveness of armaments, with consideration given to quality, reliability, and various forms of enemy counteraction. The book contains information on classic, as well as new, mathematical methods for optimization, used in solving military-technical tasks.

Particular attention is given to the statistical modelling method with electronic computers [EVM], because it has the greatest generality and universality.

The material discussed in the book is elucidated by the use of a great many examples. Tables designed to facilitate the calculations are contained in an appendix.

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UDC 519.8

The book deals with the fundamental characteristics of the equipment employed in combat engineering, as well as with the reliability, efficiency and economy of this equipment. Methods are presented for the evaluation of these characteristics in various combat and operational situations by means of analytical methods and the method of statistical modeling.

We examine problems on the evaluation of armament efficiency in various combat situations, with consideration of its quality, reliability and various forms of enemy countermeasures. Information is presented on classical and new mathematical methods of optimization employed for the solution of military engineering problems. A particularly detailed presentation is offered with respect to the method of statistical modeling on electronic computers. The material covered in this book is clarified through the use of numerous examples. The appendix to this book contains tables intended to facilitate calculation.

The book is intended for the many engineers who deal with problems of developing, testing, producing and operating combat equipment.

PREFACE

In the development, manufacture and operation of combat equipment it is necessary to resolve numerous problems to ensure maximum efficiency for stipulated expenditures or to achieve minimum expenditures for stipulated efficiency. In connection with the fact that combat equipment is becoming increasingly more complex, the solution of such problems becomes increasingly more difficult. While it was possible, prior to the Second World War, to use only the most elementary calculations based on the simplest mathematical methods in the processes of decision-making with respect to military engineering problems, the situation has now changed drastically. At the present time, to make proper decisions it is necessary to carry out numerous and laborious calculations based on various mathematical methods which have been categorized under the common heading of "methods of operations research." These methods have found extensive application in recent years in various fields of human activity — in industry, in transport, in trade and in military affairs. A new applied science is developing before our eyes and this discipline is known as "operations research."

Despite the youth of this new science, the literature in this field has become quite extensive. Many nations publish books and specialized journals, hold conferences on operations research and publish the transactions of these conferences. With regard to operations research in combat engineering, the literature in this field consists primarily of journal articles. We sense a need for books containing a systematic outline of the methods of operations research and their application to combat engineering problems. This book is an attempt to fill this gap.

At this point we should also make mention of the book by Ye.S. Venttsel' "Vvedeniye v issledovaniye operatsiy" [Introduction to Operations Research], published at the time that this book was being readied for press and which the authors recommend to the readers. Our book is intended for a wide range of engineers who are concerned with the development, testing, manufacture and operation of military hardware. The material covered in this book requires of the reader a knowledge of the general courses of advanced mathematics and the theories of probability as taught at institutions of higher technical education (on the scope, for example, of the well known book by Ye.S. Venttsel').

The book discusses the fundamental characteristics of military hardware, methods are presented for the evaluation of the effectiveness and reliability of this hardware in various combat and operational situations by means of a variety of analytical methods, and there is also a method for statistical modeling on

electronic computers. Particular attention is devoted to this last method, since it exhibits great generality and universality. The material in the book is clarified through the use of numerous examples which are illustrative in nature and are hypothetical. The appendix to the book includes tables intended to facilitate calculations. Moreover, to facilitate modeling on computers we have presented approximation polynomials for most of the tables contained in the appendix.

The book has been divided among the authors in the following manner. Yu.V. Chuyev served as the general editor; in addition he authored §§0.9, 1.11, Chapter 2, §§6.4, 6.6, Chapter 8 (with the exception of §8.7); P.M. Mel'nikov wrote §§1.3 , 1.5, Chapter 5 (with the exception of §5.3) and §§7.8, 8.7; S.I. Petukhov was responsible for the writing of Chapter 4 and §§7.3, 7.4, 7.5, 7.6 and 7.7; G.F. Stepanov wrote §§1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 3.1, 3.2, 3.7, 6.1, 6.2, 7.1; Ya.B. Shor wrote the introduction (with the exception of §0.9), §§1.7, 1.8, 1.9, 1.10, 3.3, 3.4, 3.5, 3.6, 6.3, 6.5, 6.7 and 7.2; V.I. Kuz'min wrote §5.3 and V.S. Bogolyubskiy derived the approximation polynomials for the appendix to the book.

This book represents one of the first attempts to produce a book on operations research in combat engineering; it is doubtlessly, therefore, not devoid of errors. The authors will be extremely grateful to their readers for any critical remarks which should be addressed to the publishers at the following address: *Moscow, Main Post Office, P.O. Box 693.*

The authors wish to extend their gratitude to N.P. Buslenko and D.B. Yudina for their assistance and advice in connection with the preparation of the book.

The following system of enumeration and references has been adopted in this book. The chapters are numbered successively from 1 to 8. The sections have been assigned two numerals, of which the first indicates the number of the chapter, and the second indicates the number of the section in that chapter. Many sections have been subdivided, and these subdivisions are denoted with capital Russian letters [translator's note: in this translation, these denotations have been altered to the corresponding letters of the English alphabet].

Formulas are numbered in sequence within the limits of each section. In referring to a formula of a given section, we indicate the number of that formula in parentheses: for example, see Eq. (15). When reference is made to a formula in another section, the parentheses include first the designation of the section, and then the sequential number of the formula in that section: for example, see Eq. (7.2.15), i.e., Eq. (15) from §7.2.

The figures and tables are identified in the same manner as references to formulas from other sections, but without parentheses: for example, Fig. 7.2.1, Table 7.2.3. The tables in the appendix at the end of the book are identified by number from 1 to 11.

The list of literature citations included at the end of the book is enumerated in numerical sequence, with the numbers indicated in square brackets in the text.

INTRODUCTION

§0.1. OPERATIONS RESEARCH

A large number of books and articles has been published during the past ten years on operations research. A number of countries publish specialized journals, hold conferences and symposia, and special scientific societies have been organized in connection with operations research.

What is *"operations research"?*

The literature has not yet adopted a generally accepted definition of this term. However, an examination of published works and an analysis of accumulated experience, in our opinion, makes it possible in the following manner to characterize this new scientific trend:

operations research is an applied science intended to find optimum solutions in various fields of human activity — industry, trade, combat engineering and in the art of military warfare;

operations research, as a rule, provides a quantitative basis for decision making, facilitates the making of a decision, but does not provide the actual decision; in order to make a decision it is generally necessary to resort to accumulated experience and sound thinking processes, since it is necessary to take into consideration those factors which are not easily evaluated from the quantitative standpoint so that they may be introduced into the calculation;

operations research operates with a variety of quantitative criteria of effectiveness (economic, combat, etc.) and achieves optimization of decision-making on the basis of these criteria;

operations research makes extensive use of mathematical methods — the theory of probabilities, mathematical statistics, the theory of mass service [systems] engineering, the theory of games, linear and dynamic programing, the methods of statistical testing, etc;

operations research is a part of the more general discipline of cybernetics.

The present book will deal only with the military engineering aspects of operations research. However, because of the general nature of the methods employed in operations research, much of the material covered here may prove useful as well for individuals engaged in other aspects of operations research.

§0.2. THE CONCEPTS OF QUALITY AND RELIABILITY IN MILITARY HARDWARE

As is well known, the *quality* of any piece of equipment is understood to refer to the totality of properties determining the degree of suitability of that piece of equipment in the applications for which it was designed. This applies equally to the great variety of military hardware and installations.

Thus, for example, the quality of artillery and rocket armament is determined by its range, rate of fire, firing accuracy (i.e., accuracy and firing pattern), maneuverability, jamming invulnerability, service life, safety in handling and during firing operations, faultfree operations, duration of useful life (in storage, transportation and operation), adaptability for repair, convenience and difficulty in servicing, etc.

The quality of reconnaissance and target radar stations is defined by their range of acquisition, the resolving power, jamming invulnerability, survivability, dimensions of the scanning field, the scanning rate, the accuracy of target coordinate determination, maneuverability, faultfree operation, service life, suitability for repair, convenience and difficulty in servicing, etc.

Reliability (general reliability) refers to the ability of a piece of equipment to function in a faultless manner, to exhibit long service life and suitability for repair, as well as to ensure the normal execution of assigned functions. This property is associated with the possibility of maladjustments arising in a piece of equipment during the course of its utilization [132].

It follows from this that reliability is a part of quality and is included therein. It should be stressed that in a number of cases reliability is a very significant part of quality, and at times, the most important part.

Frequently, in the place of the term "reliability," we utilize the term "operational reliability." The purpose of this is to underscore the fact that reliability is brought to light in the operation of the equipment. Thus, *operation of the equipment* is understood to refer to the totality of all phases of its existence: storage, transportation, testing, preparation for utilization as specified, utilization as specified, technical servicing and repair.

Faultless operation refers to the capability of a piece of equipment continuously to retain its readiness for operation (i.e., no breakdowns) under specific conditions of operation. *Breakdown* is understood to refer to complete or partial loss of operational readiness.

Operational readiness refers to the status of a piece of equipment in which it can satisfy all specifications, at a given instant of time, as set with respect to the basic parameters of the equipment. If a certain piece of equipment does not meet all specifications set with respect to its secondary parameters,

such equipment status is referred to as *secondary malfunction*. In this case there is no disruption of equipment operational readiness [efficiency] and no breakdown.

Service life of equipment refers to the capability of that piece of equipment to retain operational readiness [efficiency] for a prolonged period of time, provided that the required technical servicing is accomplished (and this may include various forms of overhaul). Service life is characterized either in terms of operational time, or by the number of operational cycles, or by the volume of work accomplished.

Suitability for repair with respect to a piece of equipment refers to the ease with which malfunctions can be corrected and to the maintenance of technical service life through preventive maintenance, detection and elimination of malfunctions and breakdowns. Suitability for repair is characterized by expenditures on labor, time and facilities on such operations.

In conclusion, let us stress the difference in approach to the concepts of "reliability" and "survivability."

Survivability refers to the property of a combat engineering installation to preserve its capability of carrying out its function despite combat damage. However, when we speak of reliability, we generally have in mind the normal conditions of operation, in the absence of combat damage.

§0.3. THE CONCEPT OF THE COMBAT SITUATION

In studying the efficiency of combat engineering installations we have to consider their function in various combat situatons. In this case, *combat situation* is understood to refer to the totality of the following conditions and information.

A. Data on Friendly Forces

the number of subject installations, ammunition and spare parts, tools and accessories;

types and number of other installations [pieces of equipment] with which the subject equipment interacts;

the stated combat problem and the time allotted for its execution;

the characteristic of reconnaissance facilities;

the characteristic of camouflage facilities;

the characteristic of communications and command facilities.

B. Data on the Enemy

types, number and characteristic of targets (for example, number of attacking enemy aircraft, their possible speeds, combat ceiling, damage probability, etc.);

types, number and characteristic of enemy fire and radio countermeasures;

characteristic of enemy reconnaissance facilities;

characteristic of enemy tactical operations (for example, velocities, altitudes, headings, enemy aircraft maneuvers, intervals between aircraft, formation, etc.).

C. External Conditions

time of year and day;

weather conditions (temperature, humidity, wind, illumination, dust, etc.);

conditions of terrain (level, hilly, mountainous, etc.);

location of nearest railroad stations, airfields, warehouses, etc.

In the analysis of the efficiency of combat engineering equipment, of great importance is the proper selection of several typical combat situations for which the study of efficiency must be carried out. In a number of cases it is possible to consider entire classes of combat situations differing from one another as to various parameters which may be replaced by their average values.

§0.4. THE CONCEPT OF EFFICIENCY IN COMBAT ENGINEERING EQUIPMENT

Efficiency of combat engineering equipment is understood to refer to the characteristic of the level of completion by that equipment of those functions for which it was designed. In the case of armament the term efficiency is understood to refer to the totality of characteristics for the level of execution by that armament of combat assignments for which it is intended. Armament efficiency is frequently determined in final analysis by the magnitude of damage inflicted on the enemy. In a number of cases, armament efficiency may be characterized by the magnitude of damage prevented.

In general, efficiency characteristics are functions of the purpose for which the armament was designed. For example, an antiaircraft rocket complex is intended to inflict damage on aircraft which fly within a certain altitude range. Let us compare two versions of an antiaircraft rocket complex. Of these, the more efficient will be the one which inflicts damage on aircraft with greater probability, with smaller expenditure of rockets and within a shorter period of time.

We can see from this example that efficiency is evaluated by means of a large number of quantitative criteria, including, for example: probability of target damage, average number of rockets expended to damage a single target, average cost to inflict damage on a single target, the average time spent to inflict damage on a single target, etc.

This circumstance greatly complicates comparison of efficiency between various versions of combat engineering equipment. It may turn out that according to certain criteria one of the compared versions exhibits predominant efficiency, whereas the other proves to be more efficient on the basis of different criteria.

Comparison of efficiency between various versions is complicated even further by the circumstance that the efficiency criteria depend significantly on the combat situation. Thus, for example, one version of an antiaircraft rocket complex may exhibit high efficiency in repelling the attacks of low-flying targets and limited efficiency in repelling the attacks of high-flying targets, whereas the other version of an antiaircraft rocket complex may exhibit precisely opposite characteristics.

The efficiency characteristics for combat engineering equipment depend on three groups of factors:

the characteristics of quality and reliability for that equipment;

the economic characteristics of that equipment (the cost of the equipment and its elements, labor expenditures with respect to servicing, etc.);

the characteristics of the combat situation in which the operation of this equipment is being examined.

Let us take note of the fact that the concept of efficiency is employed most frequently with respect to weapons systems and to individual pieces of combat engineering equipment which execute independent functions (for example, reconnaissance radar stations). However, the concept of efficiency is not applied to many pieces of combat engineering equipment included in weapons systems. Thus, for example, we do not speak of the efficiency of the launching installation of a rocket complex.

On the other hand, it is always possible to speak of reliability criteria for individual installations included within a system, but it does not always make sense to speak of reliability criteria for a system as a whole (see §1.10).

§0.5. SELECTION OF CRITERIA TO EVALUATE ARMAMENT [WEAPONRY] EFFICIENCY

As was pointed out earlier, armament efficiency may be characterized by a large number of various quantitative criteria. As such criteria we may employ:

the probability of completing the combat assignment within a given combat situation;

the probability of inflicting damage on a given number of targets;

the mathematical expectation of the number of damaged tar-

gets;

the mathematical expectation of ammunition expenditure on the completion of the assignment;

the mathematical expectation of cost of facilities expended on completion of the assignment;

expenditure of ammunition to ensure execution of assignment with the given probability;

mathematical expectation of expenditure of time on completion of assignment;

mathematical expectation of damage inflicted on enemy;

mathematical anticipation of damage inflicted by the enemy, etc.

Selection of criteria depends on the goal of the study being carried out, the characteristics of the weaponry versions being compared and on the target for which the armament is intended. The criteria must be sensitive to the varying parameters of the types of armament.

A clear example of the importance of proper selection of efficiency criteria is presented in Reference [50]. During the Second World War antiaircraft weapons were carried aboard British merchant vessels to fight off attacking aircraft. These measures were implemented by reducing the antiaircraft cover of other important sites and cost quite dearly. There arose a question as to the feasibility of this measure. To resolve this problem, expensive information on aerial attacks against merchant vessels was processed. It turned out that with respect to the criterion "mathematical expectation of damage inflicted on the enemy" the efficiency was low — only about 4% of the aerial attacks were concluded with the destruction of the enemy aircraft. It developed from this criterion that the firing of antiaircraft weapons from merchant vessels did not even offset the expenditure of their installation. But, in addition, a determination of the criterion "mathematical expectation of damage inflicted by the enemy" was also carried out. It turned out that (of the total number of vessels attacked) 25 per cent of the vessels without antiaircraft weapons was sunk, whereas of the vessels with antiaircraft cover only 10 per cent of those attacked were sunk. Thus, installation of antiaircraft weapons on merchant vessels reduced the damage inflicted by the enemy by a factor of 2.5, which more than offset the expense of installing such weapons on the vessels.

Since the basic purpose of installing antiaircraft facilities was not the destruction of enemy aircraft, but the protection of the vessels, the second criterion was the correct basis of procedure.

Thus we can see that the correct solution of a given problem may depend on the selection of the efficiency criterion.

Let us also take note of the following circumstance. If we are comparing two weapons systems on the basis of a given efficiency criterion, we should not lose sight of the other indicators characterizing these systems. Such indicators may be the various characteristics of quality (for example, reliability, handling safety, etc.). Having compared two systems as to efficiency, care should also be taken to make certain that the remaining indicators of quality for these systems fall within specified permissible limits.

§0.6. ECONOMIC CHARACTERISTICS OF VARIOUS SPECIMENS OF COMBAT MATERIEL AND WEAPONRY

The economic characteristics of combat materiel and weaponry may be divided into two groups:

A includes the characteristics defined by the labor input on the fabrication of the subject items;

B includes the characteristics defined by the labor input in the operation of these items.

The characteristics of group A can always be expressed in monetary units. The characteristics of group B cannot always be expressed in monetary units. Thus, for example, the labor input involved in the utilization of combat materiel and weaponry, when there is rather great likelihood of injury being inflicted on the crew servicing this equipment, cannot be expressed in monetary units.

Group A includes the following characteristics:

the cost of preparing the design draft and the cost of adjusting the experimental model;

the cost of capital construction necessary for series manufacture;

the cost of fabricating series specimens;

the cost of fabricating spare parts and accessories;

the cost of fabricating packing and transport facilities.

Group B includes the following characteristics:

the cost of transporting the goods;

the cost of storing the goods;

the cost of current expenditures required for operation of the products (including the cost of items spoiled in storage);

the cost of capital construction required for storage and operation of the materiel and equipment;

cost of training specialists to service the materiel and

equipment;

the number of people in the crew servicing these items;

labor input in man-hours for various stages of materiel and equipment operation in combat units.

§0.7. CLASSIFICATION OF COMBAT ENGINEERING INSTALLATIONS

Combat engineering installations can be divided into five large groups:

A. Facilities to inflict damage.

B. Means of delivering damage-inflicting facilities to the target.

C. Facilities to obtain and process information.

D. Facilities to control combat equipment and military actons.

E. All remaining combat engineering facilities.

As examples of the facilities in group A we can cite artillery shells, rocket warheads, aircraft bombs, etc.

As examples of the facilities in group B we can cite artillery pieces, rockets and their launching installations, bomber aircraft, etc.

As examples of the facilities of group C we can cite reconnaissance radar stations and special digital computers designed to process reconnaissance data.

As examples of the facilities in group D we can cite command electronic computers and radio communications lines.

As examples of group E we can cite radio jamming stations.

The facilities of these groups are frequently consolidated into single systems. Thus, for example, an antiaircraft rocket complex includes antiaircraft rockets, launching installations, rocket guidance stations, as well as reconnaissance and target-indication stations.

On the other hand, various combat engineering installations are broken down into individual units and elements. For purposes of the examination into the problems of reliability, of great significance is the breakdown of these items into two classes:

one-time items which, in the event of failure, need not or cannot be repaired;

re-usable items which, in the event of failure, can be re-

paired.

We should stress that the same items may be re-usable in one phase of operation and of the one-time variety in another phase.

As examples of one-time items we can cite the elements of electronic radio equipment (vacuum-tube devices, semiconductor devices, resistors, capacitors, etc.), components of machines and instruments (gears, bearings, gyroscopes, etc.), detonators, rocket warheads, fuzes, etc.

As examples of re-usable items we can cite the various electronic computers, radar stations, communications equipment, cars, rocket launchers, etc.

Let us stress that on-board rocket facilities may be regarded as re-usable in storage and during preparation for launch, whereas they must be regarded as one-time items in flight operations.

Of great significance in the analysis of problems of reliability is the separation of combat engineering facilities into yet two other classes:

facilities without reserve (without spare parts) which are characterized by the fact that malfunction of any element in such facilities results in the breakdown of the entire device;

facilities with reserve (with spare parts) for which breakdown of a number of elements does not result in the failure of the entire device.

§0.8. CLASSIFICATION OF COMBAT ENGINEERING ASSIGNMENTS

The combat engineering assignments considered in our book may be divided into three classes:

1. Assignments which arise during the course of establishing tactical engineering specifications with respect to new models of combat equipment.

2. Assignments arising in the development and testing of the experimental models of new equipment.

3. Assignments arising in the operation of series-produced items.

The first class of assignments includes:

validation of the tactical engineering specifications imposed on new specimens of combat equipment;

validation of optimum armament system.

The second class of assignments includes:

comparative evaluation of various draft versions and development of recommendations with respect to selection of the best versions;

evaluation of quality, reliability and efficiency of newly developed models on the basis of their draft data and on the basis of test results;

economic evaluation of newly developed models.

The third class of assignments includes:

evaluation of quality, reliability and efficiency of series-produced models on the basis of test and operational results;

economic evaluation of these models;

development of optimum system of technical servicing (a system of preventive maintenance operations, spare-part and accessory standards, overhaul system, etc.);

development of optimum operational regimes (norms for acceptance in various stages of readiness, norms for combat utilization based on weather conditions, based on target parameters, etc.).

§0.9. APPLICATION OF MODELING IN OPERATIONS RESEARCH

In the solution of many combat engineering assignments it proves to be advisable and efficient to employ the methods of modeling. Let us clarify the foregoing by means of an example. Given that we have to define the optimum tactical engineering characteristics of a rocket complex which is scheduled for development. Since these specifications, as a rule, are contradictory (the greater the maximum range, the greater the weight and cost; the higher the firing accuracy, the more expensive and more complex the guidance system, etc.), a quantitative basis must be provided for these specifications.

These characteristics could be validated with greater precision by the testing of a number of rocket-complex versions under actual combat conditions. However, such tests in the overwhelming number of cases are unrealistic, first of all, because of a lack of opportunity for such tests in peacetime and, secondly, because of the colossal cost involved in the development of rocket systems. In this case the most expedient method of operations research is the modeling of combat-action processes, the comparison of various solutions on these models and the selection of the best of the solutions.

There are three fundamental modeling methods: mathematical, physical and combined (see Fig. 0.9.1). Mathematical models differ from the originals in physical nature and geometric shape, but they exhibit similarity in that they are described by identical mathematical equations. Physical models are similar to the original in physical nature and geometric shape, but differ from the original in dimensions, in the speeds of the process and in

terms of other properties which are taken into precise consideration. The combined model represents a combination of the mathematical and physical model, with that portion of the process which is most difficult or does not lend itself to description by mathematical functions being modeled physically.

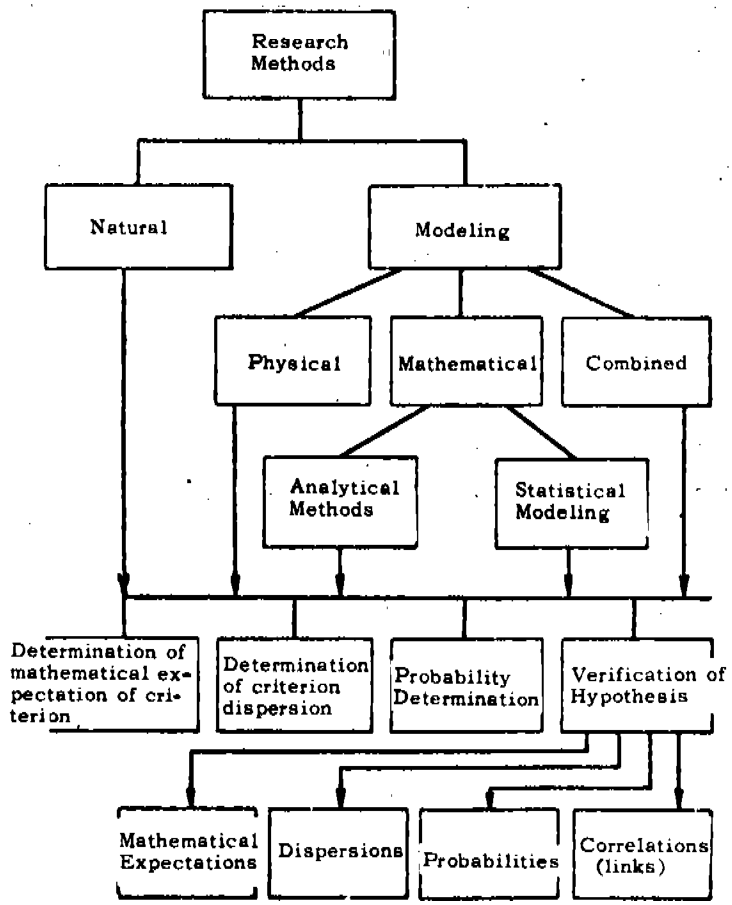

Fig. 0.9.1

An advantage of the mathematical models is the universality of the methods and apparatus for their investigation (we have reference here to the various computer devices, starting with a slide rule and ending with the most complex electronic digital computer); the possibility of studying any processes including those which cannot presently be achieved physically; the most extensive possibilities and the greatest simplicity in finding optimum solutions.

An advantage of the physical model is the possibility of studying all processes regardless of whether or not they lend themselves to description by mathematical means, and also the great clarity of results. Among the methods of physical modeling we can include special war games and studies, as well as individual test-range operations (excluding those which must, of necessity, be carried out under natural conditions). It may seem, at first glance, that these methods, which have been applied to military affairs for a long time, in any event, long before the

term *modeling* came into being, are capable of providing answers to all questions. However, this is by no means the case. An important element of combat is the counteraction of the enemy, and this process in any game or test can be taken into consideration only in an extremely conditional manner, and this thus cannot help but have an effect on the over-all results of such a modeling operation. A definite advantage of physical modeling is the participation of man, the description of whose actions in each specific situation by any given algorithms will be very difficult.

It is obvious that the best results can be achieved by a combination of mathematical and physical modeling, and this combination may involve stages, i.e., a mathematical model, then a verification of the results in special studies, followed by a refined mathematical model, or it may involve a combination of a mathematical and a physical model (for example, the incorporation of a human being into the mathematical model).

Mathematical modeling in recent times has found extensive application because of the achievements of mathematics which have made it possible to develop and investigate rather complex models, as well as because of the development of electronic computers. On the other hand, the growth in the potentials of contemporary weapons in conjunction with the rise in their cost calls for increasingly extensive application of the methods of mathematical modeling.

A mathematical model represents a system of mathematical equations and rules of logic whose utilization make it possible to calculate criterial values for each selected version with the given parameters. Mathematical models can be divided into two fundamental groups: models of statistical tests and analytical models.

The method of statistical tests involves the obtaining of a number of random realizations of a criterion and the subsequent statistical processing of these realizations.

The analytical method makes it possible to calculate the mathematical expectation of a criterion and its dispersion by means of analytical formulas.

Chapter 1

CERTAIN CHARACTERISTICS OF ARMAMENT

§1.0. INTRODUCTION

In the first chapter we consider the fundamental characteristics of armament which may be used as the initial data for the solution of problems in operations research. In certain assignments with respect to operations research we assume characteristics of armament accuracy, reliability and effectiveness as the initial data. In other assignments, we assume the cost of the armament or the effective range of the reconnaissance facilities as these initial data. Each of these characteristics may be assigned a specific value in certain assignments of operations research, whereas in others they must be estimated prior to the beginning of the research.

Determination of both accuracy and reliability characteristics for armament is rather complicated; occasionally, this is possible through experimentation. Particular attention is therefore devoted to these questions in this chapter (§§1.1, 1.2, 1.7-1.10).

This chapter also deals with damage probabilities (§§1.3-1.5). Quite a number of works have been published on this question. However, the importance of these characteristics for operations research has led to the need for a brief description.

One section is devoted to detection [acquisition] range (§1.6).

In conclusion, cost characteristics are examined in §1.11.

A number of important armament characteristics whose determination is rather simple (weight, maneuverability, etc.) are not considered.

§1.1. FUNDAMENTAL CHARACTERISTICS OF FIRING ACCURACY

A. The Fundamental Concepts of Firing Accuracy

Firing accuracy is an objective characteristic of armament quality. It is not by chance that each new form of weapon is tested prior to acceptance to determine accuracy characteristics. *Accuracy characteristics* are understood to refer to the characteristics of firing accuracy and pattern.

Firing accuracy is evaluated in terms of deviation of the

mean trajectory from the center of the target, while firing pattern refers to deviations of individual shells [missiles] from the mean trajectory.

In speaking of firing-accuracy characteristics, we should distinguish between generalized and special characteristics. The former describes firing accuracy under a great variety of conditions (time of year, weather conditions, firing azimuth, etc.). The latter are determined for specific (special) conditions such as, for example, a specific season of the year or a specific firing azimuth.

The distribution functions corresponding to averaged characteristics represent a superposition of functions corresponding to a large number of special characteristics.

Let us examine the factors responsible for deviation of rockets and missiles from their targets. The basic factors responsible for the deflection of "ground-to-ground" rockets from their targets will be the following [131]:

1) geodesic errors;

2) weather forecast errors;

3) ballistic errors;

4) errors due to technical scattering.

The scattering of artillery shells is brought about by the same factors.

Occasionally it is more convenient to divide the causes of scattering into those dependent on the rocket (with consideration of weather-forecast and ballistic errors) and the scattering which is independent of the rocket (geodesic errors and errors in the indirect laying of the line of sight). Regardless of which scattering characteristics are used, it must be borne in mind which of the factors leading to the scattering are being taken into consideration here.

Firing errors are characterized by their mathematical expectations and the standard deviations σ or their squares, i.e., the dispersions $D = \sigma^2$ [10, 24].

Frequent use is made in firing theory of the main probable deviations E (mean deviations) and, according to American literature [49], the circular probable error CPE [CKO]. The circular probable error is defined as the radius of a circle about a target, with the probability of hitting within that circle equal to 50%. This radius is occasionally denoted r_{50}.

The mean deviation is half the depth of a band of infinite width for which the hit probability is equal to 50%.

There exists a certain relationship between σ, E and r_{50} which will be shown below (see subsection B).

In actual practice we use yet another term: "maximum deviation." However, for the majority of distribution functions this term is devoid of significance if no indication is provided as to the probability that the deviation will or will not exceed this "maximum deviation."

For the maximum deviation we generally assume $\pm 4E$, and occasionally $\pm 3\sigma$.

B. The Normal Distribution Function for Firing Errors on a Plane

Firing errors are random quantities which are characterized by distribution functions. The normal distribution function for random quantities is the one most frequently employed in actual practice. The random deviation of a missile from a plane is characterized by two random quantities for the impact-point coordinates, and in space, by three random quantities; it is therefore advisable to treat the normal distribution function for firing errors separately for a plane and for space.

It is demonstrated in the theory of probabilities that this function is limited. The summation of a large number of approximately identical deviations distributed according to various functions will lead to a deviation distributed in accordance with the normal function. These conditions are generally satisfied for firing errors.

In the general case, the density of the normal distribution of firing errors in a plane is expressed by the formula

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2r(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\}, \quad (1)$$

where m_x and m_y are the mathematical expectations of random deviations in the points of missile [shell] impact along the coordinate x- and y-axes;

σ_x and σ_y are the root mean square firing deviations;

r is the correlation factor for the magnitudes of x and y.

The density function $f(x,y)$ for the normal firing-error distribution defines the probability of missile [shell] impact within the elementary area $\Delta x\ \Delta y$. It is not difficult to prove [25] that the density of firing-error distribution along one of the coordinates is also subject to the standard function having the density

$$f_1(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2}\right] \quad (2)$$

and with density

$$f_2(y)=\frac{1}{\sigma_y\sqrt{2\pi}}\exp\left[-\frac{(y-m_y)^2}{2\sigma_y^2}\right], \tag{3}$$

while the correlation factor

$$r=\frac{K_{xy}}{\sigma_x\sigma_y}, \tag{4}$$

where K_{xy} is the coupling moment.

Graphically $f(x,y)$ can be shown in the form of a hill (Fig. 1.1.1) whose apex is situated over point (m_x, m_y), while the width of the base is infinite. The steepness of the slopes of this hill is a function of the root mean square deviations σ_x and σ_y. The width of the base may be limited $\pm 3\sigma_x$, $\pm 3\sigma_y$ from the point having the coordinates (m_x, m_y).

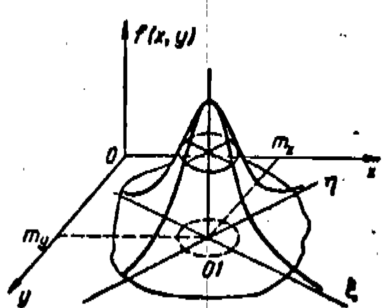

Fig. 1.1.1

A number of sections of the surface $f(x,y)$ by planes parallel to xOy will yield a family of similar and identically located ellipses which, in projection onto a plane, will have a common center at point (m_x, m_y). The probability density $f(x,y)$ is constant at all points of each of these ellipses. Such ellipses are therefore referred to as *ellipses of equal density* or *scattering ellipses*. The η and ξ axes, which pass through the major and minor axes of the scattering ellipse are referred to as *the principal scattering axes*.

If we combine the coordinate origin with the point (m_x, m_y) and the coordinate axes x and y are turned through an angle α and if these are combined with the principal axes, we will derive an equation for the scattering ellipse in canonical form. The angles α_1 and α_2 are determined from the equation

$$\operatorname{tg} 2\alpha=\frac{2r\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}, \tag{5}$$

where the angles α_1 and α_2 differ by $\pi/2$.

The canonical of the normal function on a plane has the form

$$f(\xi,\eta)=\frac{1}{2\pi\sigma_\xi\sigma_\eta}\exp\left(-\frac{\xi^2}{2\sigma_\xi^2}-\frac{\eta^2}{2\sigma_\eta^2}\right), \tag{6}$$

where σ_ξ and σ_η are the principal root mean square deviations in firing.

Generally, in processing the measurement results for deviations of missiles [shells] from a target on a plane, an effort is made to choose the coordinate axes ox and oy in advance in such a manner that they coincide with the principal scattering axes. For

this the ox-axis must be made to coincide with the firing direction [line of sight]. The root mean square deviations along the x- and y-axes will be the principal root mean square deviations in this case, and the normal function will have the form

$$f(x, y)=\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2}-\frac{(y-m_y)^2}{2\sigma_y^2}\right]. \qquad (7)$$

In the case of a two-dimensional normal distribution the probability of impact in a circle of radius r is determined from the formula

$$P(r)=\int\limits_r f(x, y)\,dxdy. \qquad (8)$$

Having substituted the density of the distribution $f(x,y)$ and having integrated Eq. (8), it is possible for us to derive the radius r_{50} of the circle within which the probability of impact is equal to 50%.

In the simplest case in which $\sigma_x = \sigma_y = \sigma$, and with $P(r) = 0.5$, we will have

$$r_{50}=r(P=0{,}5)=1{,}1774\sigma \qquad (9)$$

or

$$r_{50}=1{,}746E. \qquad (10)$$

In the more general case in which the dispersions $\sigma_x^2\neq\sigma_y^2$ are not equal, the integration and approximation of the derived function $r=f(\sigma_x, \sigma_y)$ yields the possibility of obtaining the following approximate relationship:

$$r_{50}=0{,}589(\sigma_x+\sigma_y) \qquad (11)$$

or

$$r_{50}=0{,}615\sigma_x+0{,}562\sigma_y, \qquad (12)$$

where σ_y is the greater value of these two errors.

Formula (11) yields more exact coindence (within limits of 3%) in the range $\frac{\sigma_x}{\sigma_y}=0{,}2\div 1{,}0$.

The principal probable deviations and the root mean square deviations are associated by the relationship $E=\rho\sqrt{2}\sigma$, where $\rho = 0.4769$

(this relationship is determined from the equation $\Phi(\rho\sqrt{2})=\frac{1}{2}$). (13)

C. Normal Distribution Function for Firing Errors in Space

The normal function of firing errors in space describes the scattering which results in firing long-range missiles. In general form the density of the normal distribution of firing errors in space is expressed by a rather cumbersome formula which depends

on nine parameters ($m_x, m_y, m_z, \sigma_x, \sigma_y, \sigma_z, r_{12}, r_{23}, r_{31}$). We will therefore limit ourselves exclusively to the canonical form of the density distribution for the firing errors in space

$$f(x, y, z) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right], \qquad (14)$$

where $\sigma_x, \sigma_y, \sigma_z$ are the principal root mean square quadratic deviations.

D. The Rice Distribution

In addition to the normal distribution function for firing errors, the *Rice distribution* is of great practical significance. The Rice distribution characterizes the magnitude of the miss distance for the missile with respect to the target with a shift in the grouping center in the case in which the distribution of the coordinates of the impact (explosion) points is normal.

The Rice distribution function refers to a function having the density

$$\begin{aligned} f(r) &= \frac{r}{\sigma^2} \exp\left(-\frac{r^2+\alpha^2}{2\sigma^2}\right) J_0\left(\frac{\alpha r}{\sigma^2}\right) \text{ when } r > 0, \\ f(r) &= 0 \qquad\qquad \text{when } r \leqslant 0, \end{aligned} \qquad (15)$$

where σ is the root mean square deviation of the quantity r

$$r = \sqrt{y^2 + z^2}; \qquad (16)$$

$$\alpha = \sqrt{m_y^2 + m_z^2}, \qquad (17)$$

m_y and m_z are the mathematical expectations of the random quantities y and z, respectively;

$J_0\left(\frac{\alpha r}{\sigma^2}\right)$ is a zeroth-order Bessel function.

The curves of the Rice distribution functions are shown in Fig. 1.1.2.

If α/σ is small, the Rice distribution function differs little from the Rayleigh distribution function and is described by the following equation [40]

$$f(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2+\alpha^2}{2\sigma^2}\right)\left(1 + \frac{\alpha^2 r^2}{4\sigma^4}\right). \qquad (18)$$

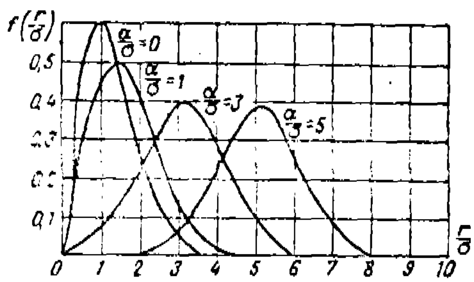

Fig. 1.1.2

The Rice distribution is occasionally referred to as the *generalized Rayleigh distribution* for reasons which will become more evident from the following subsection.

E. The Rayleigh Distribution

In the absence of systematic deviations in $\alpha/\sigma = 0$ and in the normal circular distribution of the coordinates for the point of missile impact (explosion) the missile miss distance is subje. to the Rayleigh distribution.

The density of the Rayleigh distribution is expressed by the formula

$$f(r)=\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \text{for } r>0,$$
$$f(r)=0 \qquad \text{for } r\leqslant 0. \tag{19}$$

The function $f(r)dr$ represents the impact probability of two independent random quantities distributed according to the normal function with identical parameters $\sigma_y = \sigma_z = \sigma$ into a ring bounded by two concentric circles of radii r and $r + dr$ whose cen ter is situated at the point of maximum density for the normal distribution $f(y,z)$.

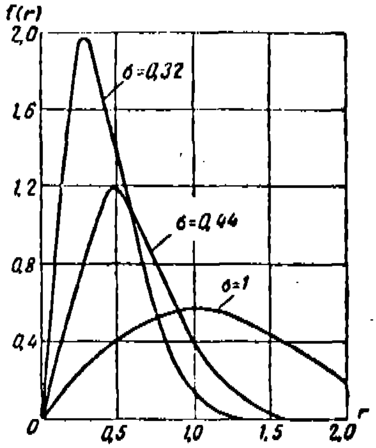

Fig. 1.1.3

Ellipses of equal density with $\sigma_y = \sigma_x = \sigma$ change into circles of radius $r = \lambda\sigma$.

We can see from Eqs. (19) and (18) that the Rayleigh distribution is a special case of the Rice distribution with the grouping center of the random quantity r made to coincide with the coordinate origin ($\alpha = 0$).

Graphically the probability den-sity of the Rayleigh distribution can be represented in the following manner (Fig. 1.1.3). The numerical character-istics of the random quantity r dis-tributed according to the Rayleigh func tion are determined from the following equations:

— mathematical expectation

$$m_r = 1{,}253\sigma, \tag{20}$$

— dispersion

$$D_r = \sigma_r^2 = 0{,}428\sigma^2. \tag{21}$$

In actual practice, the eccentricities of component parts in the machine-building industry, the service life of certain types of electronic tubes and, as has been indicated earlier, the miss distances of missiles [shells] and rockets in artillery operations are subject to distribution [82].

The Rayleigh distribution function is found from the equation

$$F(r)=\int_0^r f(r)\,dr=1-\exp\left[-\frac{1}{2}\frac{r^2}{\sigma^2}\right]. \qquad (22)$$

It is not difficult to calculate $F(r)$ in accordance with Eq. (22), but it is also possible to use tables (see Table 8 in the appendix).

F. Random and Systematic Firing Errors

Firing accuracy is associated with deviations in the explosions from the center of the target, i.e., with firing errors. What type of firing errors can there be and how are they to be classified?

Let us demonstrate this by means of examples. Let us assume that the firing operations are being conducted against a target situated on a horizontal plane, all other conditions equal. We will determine the deviations of the explosions from the center of the target. In this case a two-dimensional coordinate system is needed and it should be convenient to make this system coincide with the center of the target. One of the coordinate axes (x) may be directed along the firing line-of-sight, while the second axis (y) is set perpendicular to the former. All explosions will be situated within an area bounded by an ellipse. The deviation of the scattering center from the center of the target characterizes the accuracy or systematic firing error, while the deviation of the individual shells [missiles] from the scattering center characterizes the random firing errors or the firing pattern.

An objective characteristic of systematic errors is their mean value, derived through multiple repetitions of analogous firing operations

$$m_x=\frac{1}{n}\sum_{i=1}^{n}x_i,\ m_y=\frac{1}{n}\sum_{i=1}^{n}y_i. \qquad (23)$$

An objective characteristic of random errors is represented by their root mean square deviations σ_x and σ_y or their dispersions D_x and D_y

$$D_x=\sigma_x^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-m_x)^2,$$
$$D_y=\sigma_y^2=\frac{1}{n-1}\sum_{i=1}^{n}(y_i-m_y)^2. \qquad (24)$$

Errors in firing at a moving target will be examined on the basis of an example of an antiaircraft complex [system]. Let us assume that the target is flying rectilinearly at a constant altitude and at constant speed. The firing operation is being con-

ducted by a four-gun antiaircraft system using shells with long-range fuzes in accordance with data derived from the solution of the missile-target impact problem by means of antiaircraft fire-control instrumentation AAFCI [ПУАЗО] and from information provided by the weapons guidance station WGS [СОН].

The system operates automatically and ensures the firing of salvos at a given rate. Let us establish the deviations of each explosion from the center of the target. In this case we will need a three-dimensional coordinate system, with the coordinate origin also easily made to coincide with the center of the target. During the period of time that the target is within the firing zone the system will execute a series of salvos.

All of the shots of a single salvo will deviate from the center of the target in accordance with the errors at the output of the instrumentation complex. However, there will be a scattering of explosions for each salvo, and this will correspond to the individual errors of each of the weapons.

On the whole, all of the explosions will be distributed within a certain space around the target in a random manner. Experience shows that when firing at a nonmoving target, the geometric body containing all of the explosions will be an ellipsoid. In our example this will also be an ellipsoid, since the coordinate origin moves together with the target.

The deviation of the ellipsoid center from the target center will be governed by systematic errors, while the deviations in the explosions within the ellipsoid with respect to the ellipsoid center will be caused by random errors. However, if we now compare the explosion ellipsoid derived from firing operations carried out today with the ellipsoid derived from firing operations on another day, given identical initial data, we will find that the centers of these ellipsoids do not coincide. Consequently, the systematic errors change. In the given case the deviation of the ellipsoid centers is due to the errors in firing preparation. However, if the deviations in the scattering centers occur repeatedly, these deviations will be systematic. They may be caused either by errors in the instrumentation system or by errors in firing preparation. Thus we should distinguish between the systematic errors that are characteristic of specific conditions (daily errors, system errors, etc.).

G. Error Groups

In the case of antiaircraft firing operations random errors are generally divided into two or three groups which are associated with the firing conditions. For example, in the firing of an antiaircraft battery the random errors are divided into three groups. The division of the firing errors into groups is accomplished on the basis of their correlation functions: errors closely associated with each other are included in one group; those errors less closely associated with each other are grouped in another, etc. The first group of errors contains individual errors, i.e., the technical scattering of the shells [missiles] and the errors due to the positioning of the weapons. The root

mean square deviation σ_I is an exhaustive characteristic of the errors in the first group.

The second error group includes errors at the output of the instrumentation system (errors of dynamics) which are characterized by the root mean square deviation σ_{II}. These errors do not result in deviations of an individual missile, but for all shots of a given sequence.

The third error group includes ballistic errors and errors in meteorological firing preparation (σ_{III}) which cause deviations in all shots of a given firing operation.

When the firing efficiency of a single weapon is being evaluated by means of an independent fire-control system the firing errors are divided into two groups. In this case, the first group contains errors of technical explosion scattering and a portion of the errors at the output of the instrumentation system, which are independent of the transition from one shot to another. The second error group includes the remaining portion of the instrumentation errors, as well as ballistic errors and the meteorological errors in preparation for firing.

§1.2. DETERMINATION OF FIRING-ACCURACY CHARACTERISTICS BY EXPERIMENTATION

A. Methods for the Determination of Accuracy Characteristics

We distinguish the following methods of determining accuracy characteristics.

1. The experimental method, associated with the direct firing of shells or rockets [131] and the subsequent processing of the derived results. This method is extensively employed in cannon-type artillery.

2. The experimental-theoretical method which is understood to refer to the utilization of simplified theoretical functions as interpolation formulas to extend experimental data to other conditions.

3. The method of statistical modeling in which perturbations are determined by means of a statistical analysis of material from laboratory tests and from the processing of launch results, with the accuracy characteristics subsequently determined by means of electronic modeling of the perturbed motion of rockets in a large number of tests [35].

4. The analytical method in which the accuracy characteristics are calculated.

Naturally, the last two methods stand out because of their higher accuracy with respect to the derivation of the scattering characteristics, since they make it possible to take into consideration a considerably larger number of factors affecting the firing operation. The resulting experimental material in this

case is seemingly enriched.

Analytical expressions for an evaluation of the accuracy of the derived scattering characteristics in statistical modeling are presented in §2.3. These make it possible to evaluate the accuracy of the results obtained under specific conditions.

In §2.5 we have an example of the application of the method of statistical modeling for an evaluation of firing accuracy.

B. Determination of Firing Errors by Experimentation

Determination of accuracy characteristics through experimentation is most conveniently demonstrated by means of examples.

We will not divide the errors into groups at this time.

EXAMPLE 1. To derive scattering characteristics, let us launch rockets under identical conditions. The point coordinates are determined by the same method. As a result we obtain n pairs of measured deviations in the explosion points from the target with respect to range and direction on the plane, in an independent manner and for identical conditions. The measurement results (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) are independent systems of random quantities for which

$$m_{x_i} = m_x,\ m_{y_i} = m_y,\ D_{x_i} = D_x = \sigma_x^2,$$
$$D_{y_i} = D_y = \sigma_y^2 \text{ and } K_{x_i y_i} = K_{xy},$$

where the subscript i pertains to the measurement pair. Find the approximate accuracy characteristics and evaluate these.

Solution. The approximate values of the independent quantities m_x, m_y, σ_x^2 and σ_y^2 are defined in the same manner as in the processing of linear measurements, i.e., on the basis of the formulas

$$\left.\begin{aligned} m^*_x &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ m^*_y &= \frac{1}{n}\sum_{i=1}^{n} y_i \end{aligned}\right\}, \tag{1}$$

$$\left.\begin{aligned} \sigma_x^{*2} &= \frac{1}{n-1}\sum_{i=1}^{n} (x_i - m^*_x)^2 \\ \sigma_y^{*2} &= \frac{1}{n-1}\sum_{i=1}^{n} (y_i - m^*_y)^2 \end{aligned}\right\}, \tag{2}$$

where m^*_x, m^*_y, σ_x^{*2} and σ_y^{*2} are the experimental values of the mathematical expectations and dispersions of the measurements.

In estimating the accuracy of the resulting characteristics

we generally make use of the following formulas (see [82], pages 128 and 136)

$$\sigma_{m^*_x} = \frac{\sigma_x}{\sqrt{n}} \approx \frac{\sigma^*_x}{\sqrt{n}}, \quad \sigma_{m^*_y} = \frac{\sigma_y}{\sqrt{n}} \simeq \frac{\sigma^*_y}{\sqrt{n}}, \tag{3}$$

$$\left.\begin{aligned} \sigma_{\sigma^*_x} &\simeq \frac{\sigma_x}{\sqrt{2n-1,4}} \simeq \frac{\sigma^*_x}{\sqrt{2n-1,4}} \\ \sigma_{\sigma^*_y} &\simeq \frac{\sigma_y}{\sqrt{2n-1,4}} \simeq \frac{\sigma^*_y}{\sqrt{2n-1,4}} \end{aligned}\right\}. \tag{4}$$

Since the instant of contact is the mathematical expectation of the product resulting from the deviations of the random quantities x and y from their mathematical expectations, the approximate value of the instant of contact K^*_{xy} is determined from the formula

$$K^*_{xy} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - m^*_x)(y_i - m^*_y). \tag{5}$$

An evaluation of the accuracy for the instant of contact is conveniently given by the correlation factor whose general characteristic is determined from the equation

$$r_{xy} = \frac{K_{xy}}{\sigma_x \sigma_y}. \tag{6}$$

The experimental correlation factor

$$r^*_{xy} = \frac{K^*_{xy}}{\sigma^*_x \sigma^*_y}. \tag{7}$$

EXAMPLE 2. To determine the scattering characteristics for an antiaircraft weapon, firing operations are carried out against a nonmoving point in three-dimensional space with n shots. The coordinates of the explosion points are determined by identical methods.

From measurements of the deviations of the explosions from the target we derived $3n$ independent random quantities. Determine the approximate values of the scattering characteristics m^*_x, m^*_y, m^*_z, σ^{*2}_x, σ^{*2}_y, σ^{*2}_z, K^*_{xy}, K^*_{xz} and K^*_{yz}.

Solution. Determination of the numerical characteristics of a system of two random quantities reduced to the processing of linear measurements. Therefore, in the case of a system of three random quantities, in analogy, we can write the working formulas for the approximate values of the fundamental numerical characteristics of the system, adding another function with respect to the z coordinate to the earlier derived functions.

At the conclusion of this section we will examine the accuracy characteristics for the case of an antiaircraft system firing at a moving target. Since the target is moving, all of the accuracy characteristics will be random functions of time.

The fundamental characteristics of random functions include the mathematical expectation $m_x(t)$, the dispersion $D_x(t)$ and the correlation function $K_x(t_1, t_2)$. And if we are considering two random functions $x(t)$ and $y(t)$ at the same time, to the above-cited characteristics we have also to add the instant of contact $K_{xy}(t)$. The basic problem of processing random functions therefore involves determination of the approximate values of these characteristics.

EXAMPLE 3. An antiaircraft artillery system functioning in conjunction with antiaircraft fire-control instrumentation [AAFCI] on the basis of data from a weapons guidance station [WGS] prepares initial data for firing at an aircraft flying at a constant altitude, rectilinearly and at a constant speed. The unit has carried out n independent experiments (observations) and as a result has obtained n realizations of the random functions $x(t)$ and $y(t)$, characterizing firing accuracy (firing errors in the picture plane).

Find an estimate of the characteristics for the random function:

$$m_x(t),\ m_y(t),\ D_x(t) \text{ and } D_y(t),$$
$$K_x(t_1, t_2),\ K_y(t_1, t_2) \text{ and } K_{xy}(t).$$

Solution. Let us examine a number of cross sections of the random functions $x(t)$ and $y(t)$ for the instants of time t_1, t_2, ..., t_m and let us record the values assumed by the functions $x(t)$ and $y(t)$ at these instants of time. The section of the random function refers to the value of its random realizations at a fixed instant of time. To each of the instants t_1, t_2, ..., t_m there will correspond n values of each function. In this case, the intervals between the instants t_1, t_2, ..., t_m are selected so that it will be possible to ascertain the most significant changes in the functions and these are generally established by the speed at which the random process is photographed.

Let us assume that the functions $x(t)$ and $y(t)$ in our example have been established with an interval of 0.5 sec. The coordinate origin $t = 0$ in this case corresponds to the instant at which the target passes through the heading parameter.

The values of the functions $x(t)$ are presented in Table 1.2.1.

For each value of t_j let us calculate the mathematical expectation, the dispersion and the correlation function in accordance with the formulas presented above (1.5). The results of the calculations are given in Table 1.2.2.

The correlation function $K_x(t_j, t_q)$ in the example has been calculated only for the single value of $t_j = 0$.

TABLE 1.2.1

Values of $x_i(t)$ [g.r.] (goniometer readings)

i	t_j, sec										
	0	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0	4,5	5,0
1	2,2	2,7	3,5	4,7	6,2	7,5	7,9	8,1	7,8	7,1	6.5
2	1,9	2,7	4,3	5,3	5,8	6,3	6,7	6,9	6,9	6,7	6,7
3	1,5	2,8	3,6	3,9	4.0	4,0	4,3	5.0	5,5	5,8	5,7
4	3,0	3,4	3,5	3,7	4,3	5,2	5,9	6.0	5,9	6,2	6,7
5	1,2	2,3	3,2	3,6	3,8	3,7	3,5	3,4	3,6	4,2	5,1
6	1,2	1,9	2,1	2,1	2,0	2,2	3,0	4,0	4,3	4,1	3,7
7	5,8	6,5	7,0	7,4	7,5	7,6	7,5	7,5	7,2	7,0	6,9
8	5,1	4,8	4,6	4,9	5,8	6,8	7,2	7,1	6,7	6,4	6,4
9	3,2	3,9	4,0	3,6	3,3	3,0	3,2	3,7	4,7	5,6	5,8
10	3.4	2,3	1,2	0,9	1,1	1,8	2,5	3,5	4.4	5,0	5,2
11	6,8	6,6	6,4	6,2	5,8	5,5	5,3	5,3	5,4	5,5	5,8
12	3,8	3,3	3,1	2,9	2.5	2,3	2,3	3.5	4,7	4,9	5,1

TABLE 1.2.2

t_j, sec	0	0.5	1,0	1,5	2	2.5	3,0	3,5	4,0	4,5	5,0
$m^*_x(t_j)$ [g.r.]	3,2	3,6	3,8	4,1	4,3	4,6	4,9	5,3	5,5	5,7	5,8
$D^*_x(t_j)$ [g.r.]^2	3,3	1,7	2,6	3,1	3.7	4,4	4,2	2,0	1,7	0,8	0,9
t_q, sec	—	0.5	1,0	1,5	2	2,5	3,0	3,5	4,0	4,5	5,0
$K_x(t_0, t_q)$ [g.r.]^2	—	1,5	1,8	1,7	1,5	0.8	1,1	0,8	0,6	0,6	0,2

The random functions are processed analogously as well with respect to the second coordinate $y(t)$ for the same fixed values of t_j.

The correlation instant $K_{xy}(t_j)$ is calculated in accordance with Eq. (5) for each fixed value of t_j.

At times it is convenient to present the calculational results in the form of graphs as functions of time, with K_x given in the form of an analytical function.

Random functions are sometimes conveniently presented in the form of a canonical or spectral expansion. The methods for obtaining such expansions are described in §2.2.

The error values obtained at the output of the instrumentation system (m^*_x, m^*_y, D^*_x, D^*_y, K^*_x, K^*_y and K^*_{xy}) are used in calculating the firing efficiency of an antiaircraft system. The results obtained in this case from the calculation of efficiency will correspond to those conditions of the combat utilization of the system at which the experimental values for the errors were obtained at the output of the AAFCI [antiaircraft fire-control instrumentation] (target velocity, target altitude, heading parameter and range to target).

§1.3. FUNDAMENTAL CONCEPTS AND CHARACTERISTICS OF DAMAGE PROBABILITY

The basic function of each shot is to damage a target. This can be achieved either by striking the target (for certain types of missiles) or by the detonation of the missile at a certain distance from the target. This distance should be no larger than that established for the given missile and target. Detonation of the missile may be accomplished either by a contact (impact) fuze actuated on impact against an obstacle or by means of a noncontact fuze which provides for detonation on the basis of a given indicator (reflection of a signal from the target, on the basis of flight time, etc.).

Target damage probability is understood to refer to the conditional probability of target damage under the condition of a specified number m of missiles striking the target (in the case of missiles with contact fuzes) or under the condition that the detonation of the missile takes place at a point having the coordinates (x, y, z) (the coordinate damage probability for a missile for a noncontact fuze).

Damage probability is determined by target vulnerability and missile warhead strength (the destructive factors).

Let us examine in greater detail the coordinate damage probability and the destructive factors of various types of ammunition [warheads]. In the general case, the probability of damage is a complex function which depends both on the coordinates of the point of impact and on the characteristics of fuze actuation. Thus, for example, in the case of antiaircraft fire a necessary condition for the damaging of the aerial target by means of a high-explosive fragmentation warhead is the fact that the vulnerable elements of these targets must be covered by the region of explosive-charge inflicted damage [49]. The instant of detonation in this case must be selected with consideration given to the velocities of target motion and the destructive elements. With this purpose in mind, each warhead is fitted out with a fuze.

The fuze must thus ensure the properly timed detonation of the warhead intended to achieve the greatest possible target damage.

A. Coordinate Damage Probability

The probability of target damage as a function of the coordinates of the points of impact (detonation) of missiles is referred to as the "coordinate damage probability." The coordinate damage probability characterizes the fully determined combination of missile and target and may be treated both in a plane and in space. As an example characterizing the coordinate damage probability in a plane we can consider the damage probability for ground targets (tanks, armored vehicles, dugouts, etc.) when attacked with several high-explosive or high-explosive fragmentation shells. Mathematically, this function is expressed as an equation of the following form:

$$G_n(x_1, y_1; x_2, y_2; \ldots; x_n, y_n) = \\ = 1 - [1 - G_1(x_1, y_1)][1 - G_1(x_2, y_2)] \ldots [1 - G_1(x_n, y_n)] \qquad (1)$$

with the condition that all of the explosions having the coordinates (x_i, y_i) are independent from the standpoint of damage, i.e., in the "absence of accumulated damage" [130].

Here $G_n(x_1, y_1; x_2, y_2; \ldots; x_n, y_n)$ is the coordinate damage probability for the target attacked with n shells [missiles] and $G_1(x_i, y_i)$ is the coordinate damage probability for a firing attack involving a single *ith* shell [missile]. The coordinate damage probability of the target attacked with n missiles represents the conditional probability of target damage on condition that the missiles [shells] explode at points (x_1, y_1), (x_2, y_2), etc., to (x_n, y_n).

In the case of firing operations against aerial targets the coordinate damage probability in analogous fashion may be written in the following manner $G_n(x_1, y_1, z_1; x_2, y_2, z_2; \ldots; x_n, y_n, z_n)$, which, in the absence of accumulated damage, is expressed in terms of $G_1(x_i, y_i, z_i)$

$$G_n(x_1, y_1, z_1; x_2, y_2, z_2; \ldots; x_n, y_n, z_n) = \\ = 1 - [1 - G_1(x_1, y_1, z_1][1 - G_1(x_2, y_2, z_2)] \ldots \\ \ldots [1 - G_1(x_n, y_n, z_n)]. \qquad (2)$$

Let us assume that all n shells exploded at the same point (x, y, z). In this case, $G_1(x_1, y_1, z_1) = G_1(x_2, y_2, z_2) = G_1(x_n, y_n, z_n)$ and from Formula (2) we will obtain

$$G_n(x, y, z) = 1 - [1 - G_1(x, y, z)]^n \approx 1 - e^{-nG_1(x, y, z)}, \qquad (3)$$

where $G_1(x, y, z)$ is the conditional probability of target damage on condition of missile [shell] explosion at the point (x, y, z).

Equation (3) is also used to calculate the coordinate damage probability of a target attacked with fragmentation shells, where the number of fragments striking the target on detonation of the shell is taken as the value of n, and where $G_1(x, y, z)$ is taken as the conditional probability of target damage by a single fragment of a given weight category on condition of shell explosion at the point (x, y, z).

B. Destructive Factors of Various Types of Ammunition

a) Destructive factors of a nuclear explosion

Explosive atomic devices are based on the utilization of the atomic energy liberated virtually instantaneously with an explosive reaction [127]. Explosive atomic weapons may presently be used in the form of atomic or hydrogen aerial bombs, missiles and rockets. These weapons are intended for the destruction of various objects [sites], the destruction of combat materiel and weapons and to inflict injuries on personnel.

A nuclear explosion may inflict damage in the following manner:

light radiation;

shock wave;

penetrating radiation;

radioactive contamination of terrain.

The nature and degree of damage inflicted by all of these factors vary and are functions of target vulnerability.

Target vulnerability. Injuries inflicted on people exposed directly to the shock wave are classified as light, medium, serious and critical [95] (Fig. 1.3.1).

At the front of the air shock wave it may be assumed that to destroy conventional urban structures the excess pressure must be $\Delta p_f \approx 0.5$ kg/cm² [128].

Light radiation causes the combustion and charring of various combustible materials. On the battlefield, light irradiation of combat personnel may produce skin burns, wooden structural and material parts may burn or char, as may the paint on aircraft, tanks and similar armament items; covers and the rubber rollers of tanks and motor vehicles may burn. Storage areas for fuel and lubricants, ammunition dumps and other warehousing facilities are particularly subject to this danger [128].

The degree of damage inflicted by the shock wave and light irradiation depends on the distance from ground zero and on the TNT equivalent of the nuclear weapons.

The degree of damage exhibited by various factors of nuclear explosion as a function of the radius and of the TNT equivalent is shown quantitatively in Fig. 1.3.1 [95].

We can see from Fig. 1.3.1 that on a clear day the greatest radius of damage will be achieved by light irradiation, whereas the shock wave will produce the greatest damage in the case of poor atmospheric transparency which may reduce the radius of damage from light irradiation by a factor of two and more, depending on the transparency factor.

The damage radius in the case of penetrating radiation is a weak function of the magnitude of the TNT equivalent of the nuclear weapons, and in terms of magnitude it is smaller than the radius of shock-wave damage.

Radioactive contamination of the terrain may also inflict injuries on human beings, if these are not provided with protective equipment. The dimensions of the territory contaminated by radioactive fallout depend on the TNT equivalent of the nuclear explosion. The degree of radioactive contamination is also a strong function of weather conditions. In rain, snow and fog con-

The curves for light and medium degrees of damage due to the shock wave effect; the curve for primary alpha, beta and gamma radiation (a 400 r dose) and the curves for light and neutron irradiation.

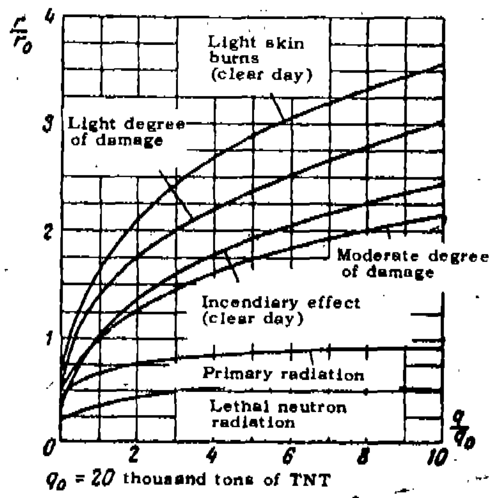

Fig. 1.3.1

tamination will be stronger. The greater the speed of the wind, the smaller the dimensions of the terrain exhiting a high level of radiation.

b) The destructive factors of chemical weapons

Poisonous substances PS [OB] and the means by which they are employed on the battlefield make up the concept of chemical weapons [129]. The basis of the destructive effect of a chemical weapon is represented by the poisonous substances which, in foreign armies, are conditionally divided into stable and unstable substances.

The stable PS include those which retain their destructive effect from several hours to several days.

The unstable PS include those substances which retain their destructive effect for several minutes, and sometimes for several hours.

Both the stable and the unstable PS inflict injuries on unprotected personnel. These substances easily penetrate structures not protected against chemical warfare, as well as tanks and other combat equipment, inflicting injuries on the personnel within such vehicles.

c) Destructive factors of ammunition with conventional explosive substances

The nature of the effect produced by ammunition of this type depends on the caliber of the ammunition and its classification (high-explosive, fragmentation and fragmentation-high explosive).

The basic destructive factor of high-explosive ammunition is the shock wave which inflicts injury on personnel and damage on materiel and weapons.

The explosive force depends on the weight of the explosive substance and on the rate of detonation, quantitatively expressed in terms of the pressure of the exploding gases at the instant of their formation, by means of the following formula [127]

$$p = \gamma \frac{D^2}{40}, \tag{4}$$

where p is the pressure in kg/m^2 at the surface of the charge;
γ is the bulk weight of the explosive substance in kg/m^3;
D is the rate of detonation propagation, in m/sec.

For TNT, according to Formula (4), we obtain

$$p = \frac{1\,600 \cdot 7\,200^2}{40} \approx 200\,000 \text{ atm.}$$

As a result of this pressure, the explosion exhaust gases begin to expand in all directions at a speed close to that of the detonation. The ambient air also begins to move at the same speed, forming an area of strong compression or an area of an air shock wave propagating in all directions from the center of the explosion at supersonic speed. The pressure at the front of the shock wave subsequently approaches the pressure of the unperturbed air and the speed of the front begins to approximate the speed of sound. The shock wave degenerates into an ordinary sonic wave.

In firing operations against aerial targets, as a result of the shock-wave effect, the structural elements of the target are compressed and other effects are produced, causing the target to loose aerodynamic stability. On the detonation of a high-explosive charge which has penetrated the target we find that the structure of the target explodes outward. For missiles [shells] of small caliber, used in firing operations against aerial targets, fitted out with contact fuzes, the basic destructive factor is the shock wave which destroys the structure, disrupts control and causes the ignition of the fuel.

An explosive shell intended to produce the largest number of fragments is referred to as a fragmentation or fragmentation-high explosive shell. The formation of fragments in this case is accompanied by a high-explosive effect which should not be overlooked in the case of explosions at short distances from the target.

However, the basic destructive effect for shells [missiles] and warheads of this type is represented by the fragments.

The fragments of artillery shells are distinguished as to shape and weight, while the fragments of antiaircraft guided missile AGM [ЗУР] warheads are approximately identical in shape and dimension (in weight) [49]. The production of such fragments is achieved by the implementation of a variety of structural [design] measures.

The characteristics of warheads (shells) achieved through the detonation of fixed warheads (missiles) on the ground (under

static conditions) and governing the effectiveness of the fragmentation effect against a target include the following: the total number of fragments and the parameters of their distribution with respect to the scattering angle within limits of $\Delta\varphi_{st}$ (Fig. 1.3.3), the weight and flight velocity of the fragments, their ballistic coefficient, etc. Distribution with respect to angle of fragment scattering under static conditions is uniform in the plane perpendicular to the axis of the warhead and nonuniform in the plane passing through this axis.

On the basis of the fragmentation characteristics derived in the detonation of warheads (shells) under static conditions and based on the conditions of encounter, we calculate the parameters of fragment distribution in motion relative to a target. The latter characteristics are used to determine the flow density and the energy parameters of the fragments which impact on the target.

The fragmentation effect against an aerial target is achieved in the form of the mechanical, incendiary and initiating effect of individual fragments or groups of fragments against the vulnerable elements of the target.

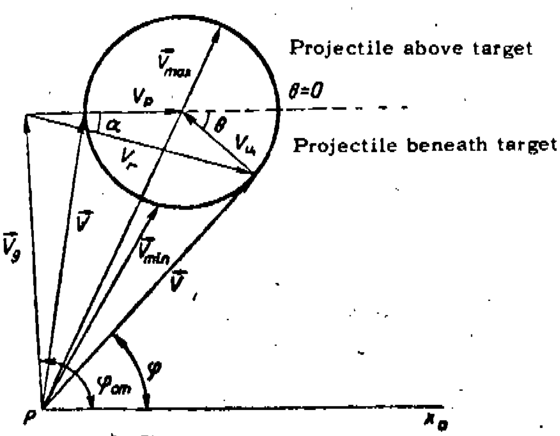

Fig. 1.3.2

The effectiveness of the fragmentation action will obviously depend on the number of fragments striking the vulnerable elements, as well as on the weight and speed of impact of these fragments relative to the target [130].

With the area of the vulnerable element determined, the number of fragments striking that area is determined by the density of the fragmentation field covering that element. Thus the efficiency of the fragmentation effect depends on the structure of the fragmentation field moving at a certain speed relative to the target. In turn, the structure and energy characteristics of the field and its orientation relative to the axis of the moving target determine the characteristics of the conditions of missile-target encounter and namely, velocity ($\bar{V}_{ts}$) and the target heading

angle (α), the velocity of the missile ($\vec{V}_r$) and the angle of its encounter with the target (θ).

As an illustration of the foregoing, in an example let us examine the manner in which the parameters of motion for a single fragment vary, i.e., let us examine the elements of the field in the case of a variation only in the angle of encounter for fixed values of all remaining characteristics. Figure 1.3.2 shows how the magnitude of V and the direction (given by the angle φ) relative to the velocity of the fragment vary as a function of the change in the angle of encounter within limits of $0 \leq \theta \leq \pi$. On detonation of the warhead under static conditions this fragment exhibited a certain velocity value — V_d — and a certain escape angle φ_{st}. The changes in direction and magnitude of fragment-target impact velocity shown in the drawing, in addition to the change in the structure of the field, govern the various values of energy and momentum transferred to the target by the fragments. As another example, Fig. 1.3.3 shows how the structure of the field of fragments moving relative to a target varies as a function of the relative velocity vector (the cases $\vec{V}_r^{(1)}$ and $\vec{V}_r^{(2)}$), with the fragments under static conditions exhibiting identical velocities V_d in the scattering sector with an apex angle $\Delta\varphi_{st}$. For these two characteristics of missile-target approach we have derived the following various field directions (φ_1, φ_2), the magnitudes for the fragment-target impact velocities ($\vec{V}_1^{(1)}$, $\vec{V}_2^{(2)}$) and the scattering sector angles ($\Delta\varphi_1$, $\Delta\varphi_2$), determining the density of the fragment stream.

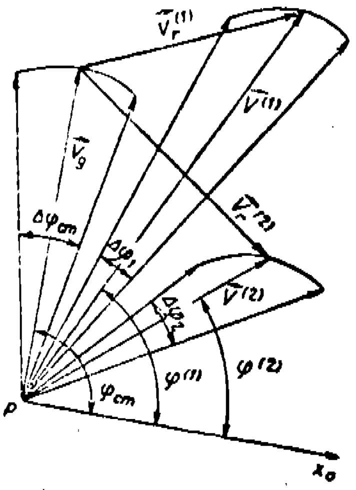

Fig. 1.3.3

The field formed by the fragments capable of inflicting damage on a target is referred to as the *region of target damage by the fragmentation effect*. This region represents a hollow cone of finite dimensions filled with the trajectories of the relative fragment motion (O_{osk} in Fig. 1.3.4). Figure 1.3.4 shows, as well, the region of the pure high-explosive effect O_f and the region of target damage resulting from the combined fragmentation-high explosive effect. This latter region is denoted $O_f \cap O_{osk}$.

The entire region O_{ts} of target damage by a fragmentation-high explosive warhead is achieved as a result of a consolidation of the regions O_f and O_{osk}.

Since we have already examined the characteristics of warhead effect, let us now turn to the problem of target vulnerability which is characterized by the vulnerability of its individual elements or "microtargets."

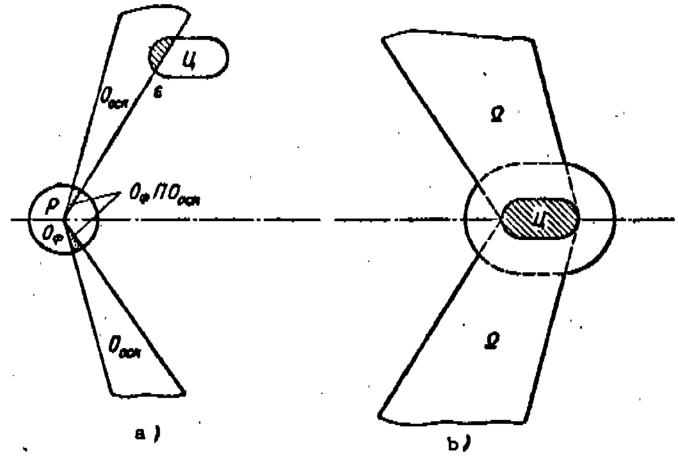

Fig. 1.3.4

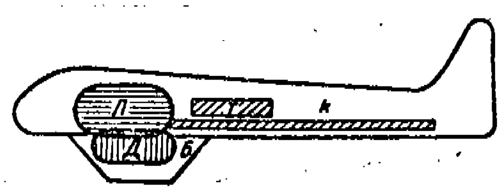

Fig. 1.3.5

According to the data in [29, 49], these basic elements of piloted aerial targets (Fig. 1.3.5) include: the flight deck Л; the engine Д; the fuel system Т; the target airframe К, the control system and the bomb load Б. On occasion, it is only the warhead which is the vulnerable element of military ballistic rockets, as is the case with the "Pershing" missile, and occasionally the vulnerable elements include the warhead, the frame and the control system ("Corporal," "Sergeant") etc. To determine the probability of damage with respect to target elements it is necessary to know their vulnerability characteristics. These include, for example, in the case of the fuel system: the vulnerable area of the system, the type and quantity of fuel in the tanks, the distribution of the tanks and fuel-feed systems to the engines within the target, the strength characteristics of the tanks and of the structural elements of the target protecting the tanks. Moreover, we require data as to the means of fuel-system protection against the incendiary effect of fragments. These include: protection of gasoline tanks with plastics which are self-sealing when penetrated by fragments, the filling of the free space in the tanks with an inert gas, special screening of the tanks, automatic fire-extinguishing equipment, etc. [29].

In the case of damage resulting from mechanical effects we must know the dimensions of the vulnerable element, its strength characteristics, etc.

Other targets are examined in analogous fashion.

§1.4. DAMAGE PROBABILITY FOR NUCLEAR WEAPONS

The damage probability for nuclear weapons is understood to refer to the probability of target damage as a function of the distance between the target and the epicenter of the explosion. Let us examine the damage probability for an atom bomb with a TNT equivalent of 20 thousand tons (the so-called nominal bomb). For this bomb in Reference [95] we find the probability of injury inflicted on personnel situated within cities as a function of the distance to the epicenter of the explosion.

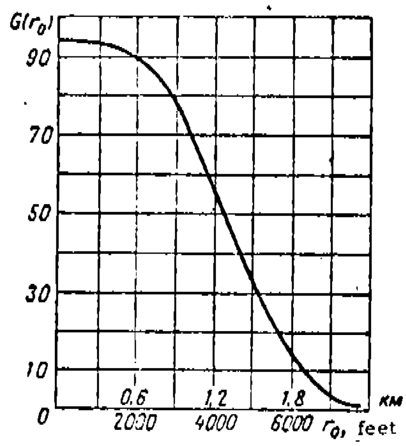

Fig. 1.4.1

Figure 1.4.1 shows the change in the percentage of extreme injury cases (fatalities) as a function of the distance from the epicenter in the case of an aerial explosion. The curve is derived for atomic bombs dropped over Japan, with a TNT equivalent of 20 thousand tons [95].

The given coordinate damage probability $G(r_0)$ will be valid for operations against protected personnel, since it is plotted on the basis of results from the effect of atomic bombs on cities. It is clear that we can thus explain why in Fig. 1.4.1 $G(r_0) \neq 100\%$ when $r_0 = 0$. We can see from Fig. 1.4.1 that at a range of up to 900 m from the epicenter there exists a high degree of injury, whereas with a range r_0 greater than 900 m the degree of injury begins rapidly to drop.

Utilizing the coordinate damage probability $G(r_0)$, we can calculate the effectiveness of bombing operations with atomic [nuclear] weapons.

For nuclear weapons different from the nominal bomb, we can calculate $G(r)$ by using the curves shown in Fig. 1.3.1, assuming that the identical damage [injury] effectiveness can be achieved at various distances from the epicenter, but with the identical effect of the shock wave or with the same energy of light irradiation.

This assumption is valid, since for nuclear weapons the law of similarity [95] which makes it possible to determine the radius of damage effectiveness for a single nuclear weapon is valid, given that this law of similarity is known for another atomic [nuclear] weapon.

For two atomic bombs, the ratio of the distances from the epicenter of the explosion at which the identical effect on the target is achieved by the shock wave and the light irradiation is proportional to the ratio of the TNT equivalent q and q_0 to a power 1/3

$$\frac{r}{r_0}=\left(\frac{q}{q_0}\right)^{1/3}, \quad (1)$$

where q_0 and r_0 are, respectively, the TNT equivalent and the distance for the bomb adopted for the purposes of the comparison (in the given case, for the nominal atomic bomb).

In Fig. 1.3.1, along the axis of abscissas, we have plotted the ratio of the TNT equivalents liberated on the explosion of the nuclear weapons, and along the axis of ordinates we have plotted the ratio of the distances from the point of the bomb explosion (the distances from the epicenter will be correspondingly smaller as a function of explosion altitude). In this case, for the comparison unit we have taken the TNT equivalent and the distance corresponding to a nominal atomic bomb. It was assumed that the energy required to produce moderate burns on the skin and incendiary effects is, respectively, equal to 3 and 10 cal/$/\text{cm}^2$. The curves in Fig. 1.3.1 have been taken from Reference [95].

The injuries inflicted on people by the shock wave directly, as indicated earlier, are divided into light, medium, serious and critical.

Figure 1.3.1 shows curves for several of these cases which are in good agreement with Formula (1) and the curves characterizing injuries resulting from neutron radiation, and these are not in agreement with the formula.

With the curves in Figs. 1.4.1 and 1.3.1 we can plot the coordinate damage probability $G(r)$ for various degrees of damage [injury].

Let us demonstrate this by means of an example:

EXAMPLE 1. Construct the coordinate damage probability for the nuclear warhead of a rocket with a TNT equivalent of 60 thousand tons with the shock wave acting on protected personnel in the case of a medium degree of injury.

Solution: the ratio $(q/q_0) = (60/20) = 3$. From Fig. 1.3.1 we find that $r/r_0 = 1.42$ or $r = 1.42r_0$. Then, from r_0 in Fig. 1.4.1 we find $G(r_0)$. Let us calculate the value of r (according to the formula $r = 1.42r_0$) and let us prepare a table for the coordinate damage probability $G(r)$.

The results of the calculation are presented in Table 1.4.1.

TABLE 1.4.1

r_0, km	0,3	0,6	0,9	1,2	1,8	2,1
r, km	0,43	0,85	1,28	1,7	2,55	3,0
$G(r)$	0.94	0.89	0.79	0.55	0.14	0.04

In operations against unprotected personnel, $G(0) = 1.0$.

§1.5. DAMAGE PROBABILITIES FOR SHELLS [MISSILES] WITH CONVENTIONAL CHARGES

Artillery shells and rocket warheads with conventional charges are used for operations both against ground and aerial targets. Depending on the design of the shell and the type of fuze, we use high-explosive, fragmentation or fragmentation-high explosive shells.

High-explosive rocket warheads are considerably less powerful than atomic warheads [131]. The radius of the damage zone for these warheads is also determined by means of Formula (1.4.1). Consequently, for these warheads we can use the same method of determining the damage probability as in the case of nuclear weapons.

In this section we will examine the determination of the damage probability for shells with a contact fuze which inflict damage only with a direct hit on the target, for fragmentation (fragmentation-high explosive) shells used against ground targets and for fragmentation warheads used in AGM [antiaircraft guided missiles].

A. Damage Probability for Shells Which Inflict Damage Only in the Case of a Direct Hit on the Target

The probability of target damage from a single hit in this case may be calculated in accordance with the following formula [10]

$$G_1 = \frac{S_i}{S} = g(S_i), \tag{1}$$

where S is the area of the projection of the target onto a plane perpendicular to the relative trajectory;

S_i is the area of the projection of the vulnerable part of the target onto that same plane.

Equation (1) is valid under the condition that the scattering exceeds the dimensions of the target and the points of impact are uniformly distributed over the area S. With this condition the frequency of striking area S_i will be S_i/S.

The probability of target damage with m hits and in the absence of damage accumulation is determined from the equation

$$G_m = 1 - [1 - g(S_i)]^m = 1 - \bar{G}^m, \tag{2}$$

where

$$\bar{G} = 1 - g(S_i). \tag{3}$$

The mathematical expectation of the number of hits (ω) required for target damage is determined from the following formula ([10], page 82)

$$\omega = \frac{n}{g(S_i)}. \qquad (4)$$

Then

$$G_m = 1 - [1 - g(S_i)]^m = 1 - \left(1 - \frac{1}{\omega}\right)^{m\frac{\omega}{\omega}}.$$

If we take into consideration that $(1 - (1/\omega))^{\omega} \approx e^{-1}$, we find

$$G_m \approx 1 - e^{-\frac{m}{\omega}}. \qquad (5)$$

We can see from Eq. (5) that the damage probability can be calculated by means of an exponential function. This damage probability is referred to as the *exponential* damage probability in the literature.

The form of the function G_m is shown in Fig. 1.5.1. We can see from Fig. 1.5.1 that theoretically G_m represents a monotonically increasing function (1). However, since m increases discretely, Curve (2) will be practically stepped.

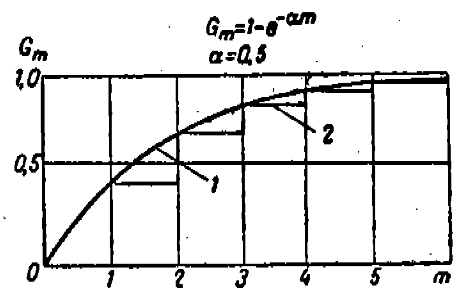

Fig. 1.5.1

The damage probability is found by an experimental-theoretical method. The experimental method is used to determine the damage vulnerability of individual parts of the target, while G_m or ω are calculated theoretically.

Let us examine the determination of ω by experimentation, following the method shown in [130]. To damage an aircraft it is necessary to penetrate the skin and to destroy or to damage the vulnerable compartments of the target. The degree of damage will vary as a function of the strength of the weapons for a given type of target. For a given type of shell, aircraft damage will occur with a single hit, whereas for another type of shell several hits will be required. This is explained by the fact that for the less powerful shells there is accumulation of damage which must be taken into consideration. For example, the first shell may penetrate the skin and the second shell, having passed through this opening, will cause the breakdown of the vulnerable compartment (control system, etc.).

For experimental determination of ω the target is conditionally divided into a number of equally vulnerable parts (com-

partments). In the event that such division is possible, some compartments of lower vulnerability are left.

As an example let us examine the calculation of ω for a hypothetical two-engine aircraft. We will assume the relative areas and damage probabilities for the individual compartments of the two-engine aircraft (with a single solid hit of a small-caliber antiaircraft shell and a certain position of the aircraft relative to the trajectory) to be the following:

TABLE 1.5.1

Aircraft or compartment parts	Relative compartment area	Probability of compartment damage with single hit
Right engine . . .	0,06	1
Left engine . . .	0,06	1
Elevator . . .	0,03	1
Right wing . . .	0,07	0,5
Left wing . . .	0,07	0,5
Fuselage . . .	0,12	0,5
Compartments damaged with a single hit (flight deck, electrical wiring, control rods, etc.)	0,46	1
Undamaged compartments	0,13	0

We can see from Table 1.5.1 that although the engine is damaged by a single hit, the probability of achieving that hit is very low (0.06); if we count on the damage of the target only with hits on the fuselage, this will require no less than four hits, since

$$G_4 = 1-(1-0{,}5)^4 = 0{,}98. \qquad (6)$$

This is high damage probability, although the area of the compartment makes up only 12% of the area of the aircraft, while the probability of four independent shots hitting this compartment is equal to $2 \cdot 10^{-4}$. On the whole, the probability of damaging this compartment will be very small ($2 \cdot 10^{-4}$). Therefore we can neglect the accumulation of damage for the fuselage and the wing center section, with the average number of hits calculated for each compartment in accordance with Formula (7), giving consideration to the probability of hitting the ith compartment

$$\omega = \frac{1}{\sum_{i=1}^{k} p_i g_1(S_i)}, \qquad (7)$$

where $p_i = S_i/S$ is the probability of hitting the ith compartment;

$g_1(S_i)$ is the probability of damaging the ith compartment with a single hit.

Formula (7) is valid for the condition that the damage of a given part or compartment of the aircraft will cause the destruction [damage] of the entire target.

By means of (7) according to the data of Table 1.5.1 we find

$$\omega = \frac{1}{0.74} = 1.4.$$

An analogous calculation of ω is accomplished for several positions of the target with respect to a relative trajectory, and these are then averaged. The resulting value of ω for each type of aircraft will be a function of the weight of the explosive charge EC [BB] (q_{BB}) for fragmentation-high explosive shells and a function of shell weight for fragmentation shells.

Contemporary antiaircraft shells exhibit the following sequence of shell- and EC-weight magnitudes [115]:

TABLE 1.5.2

Projectile type and caliber (mm)	Projectile weight kg	Explosive charge weight, kg
20 (semiarmorpiercing)	0.138	0.005
20 (fragmentation)	0.120	0.015
30 "	0.420	0.060
40 "	0.960	from 0.115 to 0.140
57 "	3.0	0.400
75 "	6.0	0.600
88 "	9.0	0.900
90 "	10.0	1.0
120 "	22.0	2.3

If we calculate the average number of hits required for each shell caliber, we can construct the relationship

$$\omega = f(q_{BB}).$$

The typical form of this function for an aerial target is given in Fig. 1.5.2. Calculations show that having established the function for a given type of target it is possible to determine a similar curve for another type of target in accordance with the equation

$$\omega = Cf(q_{BB}),$$

where C is determined experimentally.

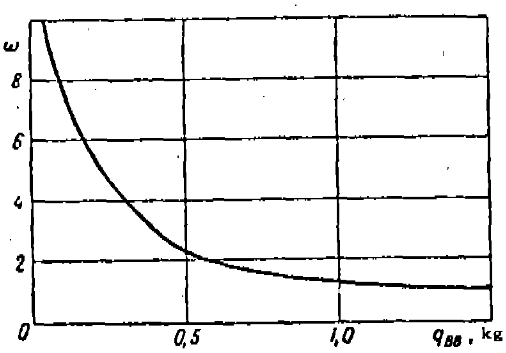

Fig. 1.5.2

The same method may be employed to determine the damage probability when using impact shells against tanks, armored vehicles and similar targets.

B. Damage Probability for Fragmentation Shells Used Against Ground Targets

Unlike damage probability for shells with impact fuzes for which the conditional probability of target damage was taken as the quality criterion, in the given case the quality criterion is assumed to be the mathematical expectation of the number of damaged targets resulting from the explosion of a single shell ([10], pages 279 and 305)

$$a_S = S(1 - e^{-nC}), \tag{8}$$

where S is the number of targets (for example, riflemen in a group);

n is the fragment density in the target (the number of fragments per square meter of area on a spherical surface of given radius with the center at the point of explosion);

C is the coefficient by means of which we take into consideration the percentage of penetrating fragments.

TABLE 1.5.3

q, g	Percentage of fragments of weight q and more	q, g	Percentage of fragments of weight q and more	q, g	Percentage of fragments of weight q and more
1	100,0	10	29,0	19	16,0
2	75,0	11	27,0	20	15,0
3	62,5	12	25,5	30	11,5
4	53,5	13	23,0	40	10,0
5	47,0	14	22,0	50	8,5
6	42,0	15	20,5	75	6,5
7	38,0	16	19,0	100	4,5
8	34,5	17	18,0	125	3,0
9	31,5	18	17,0	150	1,5

For the determination of C in accordance with Table 1.5.3 the minimum weight q of the fragments which will penetrate must be determined experimentally and the percentage of fragments of a given weight and higher must be taken as C.

Equation (8) is applicable for an evaluation of the mathematical expectation of the number of damaged targets both when firing rockets with a built-in warhead and when firing artillery shells so that they will ricochet, or when firing delayed-action high-explosive shells. For each of the enumerated types of shells the density of penetrating fragments and their percentage of the total number of fragments are determined experimentally.

Using the delayed-action shell as an example, let us demonstrate the calculation of the required characteristics.

The number of fragments and their weight depend on the caliber of the shell, the quality of the metal, the weight and grade

of the explosive charge. The weight distribution of the fragments is determined experimentally ([10], page 293).

Table 1.5.3 shows the distribution of the fragments by weight on the explosion of a steel delayed-action high-explosive shell as a percentage of the total fragment weight.

The flight velocities for the fragments vary greatly.

The initial velocity of the fragment may be calculated from the formula

$$v=\sqrt{v_з^2+v_c^2+v_зv_c\cos\beta'}, \qquad (9)$$

where v_z is the velocity imparted to a fragment by the explosive charge;
v_s is the velocity of the shell at the instant of the explosion;
β' is the angle between the directions of the velocities v_z and v_s.

The great variety of fragment shapes leads to various losses in fragment flight velocity. This circumstance makes it impossible to establish a constant magnitude of the penetration interval for all fragments.

TABLE 1.5.4

Impact Interval in Meters

q, g	v, m/sec			
	400	800	1 200	1 400
1	2,3	4,1	5,4	5,9
5	8,2	11,2	13,4	14,4
10	13,0	16,8	19,6	20,8
20	18,0	22,8	26,2	27,8
50	29,3	35,9	40,6	42,8
100	41,1	49,3	55,2	58,1
200	57,1	67,5	75,1	78,6

The penetration interval is generally regarded as the distance of the explosion from the target at which half of all fragments exhibit a kinetic energy adequate to damage the target. The penetration interval is determined experimentally. Table 1.5.4 gives the numerical values of the penetration intervals for various initial velocities and for various weights q of the fragments. In this case the energy needed to damage the target (to injure personnel) is assumed to be equal to 10 kg-m.

To determine the nature of fragment dispersal we examine the surface of a sphere with its center at the point of shell explosion. The surface of the sphere may be conditionally divided into 19 belts of 10° each (Fig. 1.5.3).

Experimental data [10] show that the relative number of fragments incident on each spherical belt will be as shown in Table 1.5.5. Here, however, we find the density of impact for n' fragments in the spherical belts on the explosion of a steel

shell (R = 3 m, N = 1000).

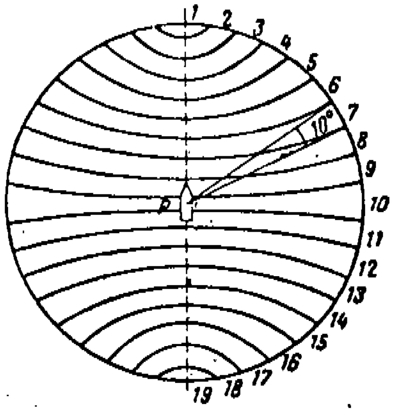

Fig. 1.5.3

Since the area of the sphere surface varies in proportion to the square of the radius, the fragment density from Table 1.5.5 must be multiplied by the ratio of the squares of the radii R and r, where R = 3 m (for which Table 1.5.5 has been compiled) and r is the distance of the target from the point of explosion.

For example, the fragment density of the 5th belt at a point removed by 10 m from the point of explosion will be equal to

$$n'_S = 2{,}8\,\frac{3^2}{10^2} = 0{,}25.$$

The fragment density for another number of fragments (proportional to the change in the number of fragments) varies analogously.

With explosion of the shell on the trajectory the fragment distribution over the spherical belts will vary as a result of a translational velocity. The fragment density in this case may be calculated in accordance with the following formula ([10], page 305)

$$n = n'\,\frac{\sqrt{(v_p^2 + v_c^2 + 2v_p v_c \cos\beta')^3}}{v_p^2\,(v_p + v_c\cos\beta')}, \qquad (10)$$

where

$$\beta' = \beta + \arcsin\left(\frac{v_c}{v_p}\sin\beta\right); \qquad (11)$$

β' is the angle between the directions of the velocities v and v_r

$$\bar{v} = \bar{v}_p + \bar{v}_c.$$

These characteristics are used for the calculation of a_S.

Let us demonstrate the method for the calculation of a_S by means of an example.

TABLE 1.5.5

Belt number	β', degrees	Relative number of fragments	Belt area, m^2	Fragment density n'
1	0	0,2	0,21	9,6
2	10	0,3	1,71	1,8
3	20	0,6	3,37	1,7
4	30	1,1	4,92	2,2
5	40	1,8	6,34	2,8
6	50	2,7	7,55	3,6
7	60	4,0	8,54	4,7
8	70	6,2	9,26	6,7
9	80	15,0	9,70	15,3
10	90	24,0	9,86	24,4
11	100	18,4	9,70	19,1
12	110	9,2	9,26	9,9
13	120	5,4	8,54	6,3
14	130	3,5	7,55	4,6
15	140	2,6	6,34	4,1
16	150	1,9	4,92	3,9
17	160	1,3	3,37	3,9
18	170	1,0	1,71	5,8
19	180	6,8	0,21	37,2

EXAMPLE 1. Determine the mathematical expectation of the number of injured targets (individual riflemen with a vulnerable surface of 0.5 m^2), if the direction to the target forms an angle of $\beta = 60°$ with the direction of shell trajectory to the point of explosion and if the distance to the target from the point of the explosion is equal to 20 m. The number of fragments weighing 10 g and more is 1000, $v_r = 1000$ m/sec and $v_s = 500$ m/sec.

Solution 1. Let us determine the density of fragments n' corresponding to a spherical surface $R = 3$ m. From Formula (11) let us determine the angle

$$\beta' = \beta + \arcsin\left(\frac{v_0}{v_p}\sin\beta\right) = 85°.$$

From Table 1.5.5 we find that $n' = 19.8$.

2. From Formula (10) let us determine n

$$n = n'\frac{\left(\sqrt{v_p^2 + v_c^2 + 2v_p v_c \cos\beta'}\right)^3}{v_p^2\,(v_p + v_0\cos\beta')} =$$

$$= n'\frac{(\sqrt{1\cdot 10^6 + 25\cdot 10^4 + 2\cdot 10^5\cos 85})^3}{1\cdot 10^6\,(1\cdot 10^3 + 500\cos 85)} = 26.3.$$

3. From Table 1.5.4, given $v = 1000\,(1.29)^{-2} = 1135$ m/sec and given the magnitude of the penetration interval ($r = 20$ m) equal to the distance of the target from the point of explosion, let us find the required fragment weight:

$$q = 10 \text{ g and more.}$$

The per cent of fragments weighing 10 g and more is found from Table 1.5.3 (29%).

4. The density of fragments at a distance of 20 m from the point of explosion will be equal to

$$n=26{,}3\ \frac{3^2}{20^2}=0{,}59.$$

5. The number of penetrating fragments incident on the area of the target is equal to

$$\bar{n}C=0{,}59\cdot 0{,}29\cdot 0{,}5=0{,}086.$$

6. The mathematical expectation of the number of injured [damaged] targets is determined from Formula (8): $a_S = 1 - e^{-nC} = 0.086$.

C. Damage Probability for AGM [antiaircraft guided missile] Fragmentation Warheads

Calculation of the damage probability for a fragmentation warhead is carried out for specific points in a region of dangerous explosions for established conditions of encounter between an antiaircraft guided missile and a target, approximately in accordance with the following scheme. Initially the parameters of the damage region (the fragmentation field) are determined. On the basis of these parameters, for each weight group (k) of fragments, we calculate the density of the fragment stream (λ_k) and the energy characteristics (E_k): the energy, momentum, etc.

Then we determine the area (S'_j) of the projection onto a surface perpendicular to the direction of the fragment stream of that portion (ω_j) of the volume (Ц) of the jth vulnerable element which is covered by the fragment field

$$\omega_i = Ц_j O_{\text{оск}}.$$

The mathematical expectation of the number of fragments of the kth group striking the jth element is equal to the product $\lambda_k S'_j$. In this case the coordinate probability for the jth element can be calculated from Formula (1.3.3)

$$G_j(x, y, z) = 1 - e^{-m_j(x,y,z)}, \tag{12}$$

where $m_j(x, y, z)$ is the mathematical expectation of the fragments damaging the target element. It is equal to

$$m_j = S'_j \sum_k \lambda_k P_j(E_k), \tag{13}$$

where $P_j(E_k)$ is the probability of element damage by a single fragment, determined on the basis of the value of the energy parameter E_k for a specific form of fragment damage effect.

EXAMPLE 2. Let there be given a vulnerable target element area $S'_j = 0.2\ m^2$ and two groups of fragments with densities $\lambda_1 = 2^{osk}/m^2$ and $\lambda_2 = 4^{osk}/m^2$, respectively, and fragment damaging-effect probabilities of $P_j(E_1) = 0.5$ and $P_j(E_2) = 0.3$. Find the

damage probability of the vulnerable element.

Solution. Having utilized Formulas (12) and (13), let us find

$$G_j(x, y, z) = 1 - e^{-0.2(2\cdot 0.5+4.03)} = 0.36.$$

The probability values of P_j, as indicated earlier, are determined experimentally; the function of a random number of fragment hits on a target is established in the same manner, etc. However, the direct reproduction of the effect of fragments on a target at a specific altitude is possible only by means of special antiaircraft rocket control ARC [3PK] tests [49]. Because of their high cost, the number of experiments is limited and does not provide sufficient information. The parameters of the damage probabilities of vulnerable elements are therefore established from the results of modeling the process of the effect of the damaging factors on target elements under ground conditions.

The coordinate damage probability for an AGM target is determined on the basis of the coordinate probabilities for its vulnerable elements. It is given by the probability of the occurrence of a specific combination of corresponding elementary events resulting in the damage of the vulnerable elements. Let us consider certain variants, introducing identical denotations for the vulnerable element and the event concluding in the damage of that element.

1. The target has been damaged to a certain extent, i.e., it is changed to this condition as a result of damage inflicted on at least one of two elements such as, for example, Л and Д. The damaging of the target as an event E [C] will be a combination representing the consolidation of concurrent events Л and Д. For independent elementary events we will obtain

$$P(\text{C}) = 1 - [1 - P(\text{Л})][1 - P(\text{Д})]. \tag{14}$$

By induction, changing to the case of n events and introducing the denotations for the damage probability, we will obtain

$$G(x, y, z) = 1 - \prod_j [1 - G_j(x, y, z)], \tag{15}$$

where G_j is the coordinate damage probability of the jth vulnerable element.

EXAMPLE 3. Particular values are given for the coordinate damage probability of the flight deck: $G_{\text{Л}} = 0.4$; for a single engine $G_{\text{Д}} = 0.2$ and for the control system $G_{\text{У}} = 0.1$ at a point having the coordinates x', y' and z'. We know that the target will be damaged if only one of these of its elements is damaged. Determine $G(x', y', z')$.

Solution. From Formula (15) we find

$$G(x', y', z') = 1-(1-0{,}4)(1-0{,}2)(1-0{,}1) \approx 0{,}43.$$

2. Let a target consist of a flight deck and two engines Д₁ and Д₂, and let its damage be achieved by damage of the element Л or of both elements Д₁ and Д₂. Assuming that the damage of each element is an independent event, we obtain an expression for the damage probability of the target:

$$G(x, y, z) = 1-(1-G_{Л})(1-G_{Д_1}G_{Д_2}). \qquad (16)$$

Here we also assume the right-hand portion of Expression (16) to be a function of the coordinates which, for the sake of simpler notation, have been dropped.

Assuming the conditional damage probabilities of uniform vulnerable elements of the *jth* type to be identical,

$$G_{Д_1} = G_{Д_2},\ G_{Л_1} = G_{Л_2} \text{ etc.}$$

by induction for the case of damage of at least one of several groups of uniformly vulnerable elements of the *jth* type we find the expression

$$G(x, y, z) = 1-\prod_{j}(1-G_j^{n_j}), \qquad (17)$$

where n_j is the total number of uniform elements of the *jth* type which must all be damaged simultaneously.

EXAMPLE 4. Given that the damaging of some targets with a high degree of certainty is possible only in the event that both pilots or both identical engines, or the fuel compartment, or the structure will be damaged or injured simultaneously. The conditional element damage probabilities at the point x', y', z' are given:

$$G_{л} = 0{,}5;\ G_{д} = 0{,}4;\ G_{т} = 0{,}2;\ G_{к} = 0{,}3.$$

Find the coordinate probability of target damage.

Solution. Considering that $n_l = n_d = 2$ and $n_t = n_k = 1$, from Formula (17) we find

$$G(x', y', z') = 1-(1-0{,}5^2)(1-0{,}4^2)(1-0{,}2)(1-0{,}3) \approx 0{,}65.$$

3. For the damage of a target to a certain extent, the flight deck Л must be damaged or no less than three of the four engines Д₁, Д₂, Д₃ and Д₄. It is assumed that the events leading to the damage of each element are independent of each other. Let us compile the various possible combinations of engine damage: $G_1 = G_{Д_1}G_{Д_2}G_{Д_3}G_{Д_4}$ represents the damage of all four engines.

$$\left.\begin{aligned} G_2 &= (1-G_{Д_1})G_{Д_2}G_{Д_3}G_{Д_4} \\ G_3 &= G_{Д_1}(1-G_{Д_2})G_{Д_3}G_{Д_4} \\ G_4 &= G_{Д_1}G_{Д_2}(1-G_{Д_3})G_{Д_4} \\ G_5 &= G_1G_2G_3(1-G_4). \end{aligned}\right\}$$

Damage to three and no damage to one of the engines

Since the combination of engine-damage events does not occur simultaneously, the probability of achieving the failure of no less than three engines is equal to $\sum_{\nu=1}^{4} G_\nu$. We will then find

$$G = G_Л + \sum_{\nu=1}^{4} G_\nu - G_Л \sum_{\nu=1}^{5} G_\nu. \tag{18}$$

Then, in view of the independence of events D_i we will have (in the denotations of damage probability)

$$\sum_{\nu=1}^{5} G_\nu = \prod_{i=1}^{4} G_{Д_i} + \sum_{i=1}^{4} (1-G_{Д_i}) \sum_{k \neq i}^{3} G_{Д_k}.$$

Having taken into consideration these substitutions into Formula (18), after elementary transformations we derive the expression for the probability of target damage in the form of

$$G(x, y, z) = 1 - (1 - G_Л)\Big[1 - \prod_{i=1}^{4} G_{Д_i} - \sum_{i=1}^{4} (1 - C_{Д_i}) \prod_{k \neq i}^{3} G_{Д_k}\Big]. \tag{19}$$

EXAMPLE 5. In the assumption of the situation just described, particular values of the coordinate damage probabilities are given for the vulnerable elements: $G_Л = 0.5$; $G_{Д_1} = G_{Д_2} = 0.7$; $G_{Д_3} = G_{Д_4} = 0.5$. Find the values of the target-damage probability $G(x', y', z')$.

Solution is obtained with Formula (19)

$$G(x', y', z') = 1 - (1 - 0{,}5)[1 - 0{,}7^2 - 2(1 - 0{,}7)\,0.7\;0{,}5^2 - 2(1 - 0{,}5)\,0{,}5 \cdot 0{,}7^2] = 0{,}74.$$

Having calculated $G(x, y, z)$ for a number of values of the coordinates for the point of shell explosion, we derive the coordinate damage probability for the target.

§1.6. RANGE OF TARGET ACQUISITION AND ITS CHARACTERISTICS

A. The Concept of Acquisition

The most important property of each form of armament which must be considered in any models of combat is the capability of that weapon [armament] to acquire [detect] a target and to reveal itself [sic]. Since the acquisition of the ith target at a

given range is a function of many random factors, the acquisition range will be a random quantity for each *jth* facility. Thus we must examine the probability of detecting the *ith* target by means of the *jth* facility.

Detection may be accomplished by visual, optical, radio-engineering and sound-measuring techniques.

In this section we will consider the probability of detecting an aerial target by means of radar and the visual detection of a ground target. We can consider both the typical case of the probability of spotting a tank or an antitank installation on the ground or the detection of ground targets from a reconnaissance aircraft. The latter case pertains to the theory of search with which we will not deal at this time.

In carrying out combat operations the troops adopt all camouflage measures in order to prevent their detection until they are ready to resort to arms. In that event, the instant of detection will coincide with the instant at which fire is opened. For all intents and purposes, many targets on the battlefield can be detected only at the instant at which they begin to perform their function.

The probability of detection depends on the range and on the status of the facility (in operation en route, under cover, etc.), on the time of day and on the external conditions (weather, relief). Formulas for the calculation of detection probability must therefore take these conditions into consideration.

B. Visual Detection of Targets

The maximum range for the detection of ground targets depends on the relief of the terrain. With defilade angles $\varepsilon \leq 0$ the maximum range is governed by the direct visibility, whereas with defilade angles $\varepsilon > 0$ the range is governed by the distance to the defilade. When firing at moving targets it is therefore convenient to regard the detection probability $P(x)$ as the product of the direct-visibility probability (Π) and the detection probability $p_0(x)$, given the condition that the direct visibility is ensured:

$$P(x) = \Pi p_0(x). \tag{1}$$

The direct-visibility probability is defined as the ratio

$$\Pi = \frac{\sum_{i=1}^{m} \Delta x_{v_i}}{\sum_{j=1}^{n} \Delta x_j + \sum_{i=1}^{m} \Delta x_{v_i}}, \tag{2}$$

where Δx_{v_i} is the visible *ith* segment of the target heading;

Δx_j is the hidden *jth* segment of the target heading;

m is the number of visible segments;

n is the number of hidden segments.

The probability of direct visibility is calculated by means of a topographic map. Terrain profiles are constructed for this purpose within the search sector in several directions from the observation point and by means of Eq. (2) these profiles are used to calculate the probability of direct visibility for moving targets (tanks, armored vehicles, etc.). Isoprobability curves for the probability of direct visibility are then plotted within the sector and these are used to calculate the probability of target detection with the aid of Eq. (1).

If we consider the detection of a moving target (a tank, a motor vehicle) under the condition that direct visibility is assured ($\Pi = 1$), it turns out that this is a random quantity which is a function of the observer, of the illumination of the terrain and of the extent to which the coloration of the target has been adapted to the coloration of the surrounding area. The basic numerical characteristics of detection range in this case include: the average detection range $Д_0$, the dispersion $\sigma^2_Д$ and the probability $P_0(Д)$ of target detection as a function of the range $Д$. All of these characteristics are determined experimentally.

As one of the possible versions, a test to determine the visual-detection range of a tank can be carried out in the following manner. The tank executes no less than ten starts in a given direction and at the same speed in each case. The tank is tracked by a reconnaissance radar station at which the range indicator is photographed, including the electrical signals transmitted to the camera from the observers. The observers are situated within the area of the radar station. The processing of these observations makes it possible to derive the density of detection-range distribution, the average range value and its dispersion, as well as the frequency of target detection as a function of range.

EXAMPLE 1. Hundred (100) measurements of range ($Д$) have been carried out on a tank by means of visual observation. The measurement results have been reduced to a statistical series (Table 1.6.1). Determine the numerical characteristics of tank detection range.

TABLE 1.6.1

$Д_i$, km	1.5; 2.5	2.5; 3.5	3.5; 4.5	4.5; 5.5	5.5; 6.5	6.5; 7.5	7.5; 8.5
m_i	4	18	33	22	19	3	1
P^*_i	0.04	0.18	0.33	0.22	0.19	0.03	0.01

Here $Д_i$ represents the limits of the range categories;

m_i is the number of values referred to each ith category;

P^*_i is the corresponding frequency, defined by the equation

$$P^*_i = \frac{m_i}{n}; \tag{3}$$

n is the number of observations.

The numerical characteristics of detection range can be determined approximately from the equations

$$Д^*_0 = M^*[Д] = \sum_{i=1}^{k} \bar{Д}_i P^*_i, \tag{4}$$

$$\sigma_Д^{*2} = D^*[Д] = \sum_{i=1}^{k} (\bar{Д}_i - Д_0^*)^2 P^*_i, \tag{5}$$

where $\bar{Д}_i$ is the average value of range in the ith category;

P^*_i is the frequency of the ith category;

k is the number of categories.

In our example:

$$Д^*_0 = 4.51 \text{ km}, \ \sigma_Д^{*2} = 1.65 \text{ km}^2, \ \sigma^*_Д = 1.28 \text{ km}.$$

The sign * indicates that the characteristics are selective rather than general.

The frequency of target detection as a function of range is defined by the statistical distribution function $P_0^*(Д)$.

It should be borne in mind that

$$P^*_0(Д_{k+1}) = \sum_{i=1}^{k} P_i = 1$$

for $k \to \infty$.

We have

$Д_k$, km	2	3	4	5	6	7	8
$P^*_0(Д_k)$	1.00	0.96	0.78	0.45	0.23	0.04	0.01

Hence we can see that reliable tank detection occurs at ranges below 3 km. With consideration of terrain relief, the probability of tank detection is determined from Eq. (1).

C. Target Detection by Means of Radar

The range of target detection by means of radar is also a random magnitude, since among a large number of factors on which it is dependent, many are random magnitudes. For example, the magnitude of the effective reflecting surface of the target, noises in the radar receiver, etc., are random magnitudes affecting detection range. Target detection range is therefore associated with the probability which is a function of target height, type of aircraft, radar-unit characteristics, means of target searches and the range to the target. Target detection probability at a given altitude can be determined experimentally according to detection frequency. However, in order to find the relationship between detection probability and conditions of target flight and the conditions of the search, a large number of experiments have to be carried out. To reduce the number of experiments and to achieve more complete search characteristics for the radar unit, analytical methods of calculating the probability of

target detection are needed.

The maximum target detection range x_{max} is determined from the condition that the power of the reflected signal, applied to the input of the receiver, is equal to the threshold power $P_{pr\ min}$ of the receiver [70].

The range equation as a function of the structural parameters of the radar station with consideration of reflection from the ground or from water and the absorption of radiowaves in the atmosphere for target elevations $\varepsilon < \varepsilon_0$, where $\varepsilon_0 = (\lambda/4h)$, in Reference [70] has the form

$$x_{max} = \sqrt[8]{\frac{G_{пер} P_{пер} S_э S_{пр} 16\pi^2}{P_{пр\ min}} (hH)^4 e^{-0,58\delta x_{max}}}, \tag{6}$$

where P_{per} is the power emitted by the transmitter, w;

G_{per} is the directivity factor of the transmitter antenna,

$$G_{пер} = \frac{4\pi}{\lambda^2} S K_{ип}, \tag{7}$$

S is the antenna aperture area, m^2;

K_{ip} is the area utilization factor for the antenna.

With a parabolic reflector having diameter $Д_1$

$$G_{пер} \approx 0,5 Д_1^2,$$

S_{pr} is the effective area of the receiving antenna

$$S_{пр} = S K_{ип}.$$

For antennas with parabolic reflectors $S_{pr} = 0.5Д_2$.

$P_{pr\ min}$ is the receiver sensitivity, w;

λ is the working wavelength, m;

S_e is the effective target reflecting surface, m^2;

δ is the radiowave attenuation in the atmosphere, db/km;

h is the antenna height, m;

H is the target altitude, m.

For elevations $\varepsilon > \varepsilon_0$ Eq. (6) has the form

$$x_{max} = \sqrt[4]{\frac{G_{пер} P_{пер} S_э S_{пр}}{16\pi^2 P_{пр\ min}} \left[4 \sin^2 \frac{2\pi hH}{\lambda x_{max}}\right]^2 e^{-0,115\delta x_{max}}}. \tag{8}$$

We can use Eqs. (6), (7) and (8) to determine x_{max}, if the basic parameters of the radar unit are known.

The curves showing δ as a function of λ, given in [70], show that when λ = 10 cm, δ = 0.

Radiowave attenuation attains a maximum (δ = 10 db/km) when λ = 0.5 cm and rapidly drops to 0 when λ = 10 cm. In most cases the attenuation of the waves in the atmosphere must be taken into

consideration on waves on the order of 3 cm and shorter.

S_e is determined experimentally. The values of S_e from the data of [25] are the following:

for heavy bombers S_e = 100-150 m^2;

for medium bombers S_e = 40-70 m^2;

for interceptors S_e = 5-15 m^2;

for the nose cone (the warhead) of an intercontinental ballistic rocket S_e = 0.2-0.5 m^2.

The wavelength λ, the transmitter power P_{per}, the receiver sensitivity $P_{pr\ min}$ and the antenna height h are determined by the structural features of the station.

The maximum detection range x_{max} would be obtained in the case of the stable functioning of all station parameters and with constant search conditions. However, since these parameters are scattered with respect to time, in actual practice we note a scattering of the detection range. Consequently, x will be a random quantity, and the event occurring in target detection at a range x will also be random. The probability of this event is associated with the structural features of the station and the search conditions. To derive the target detection probability as a function of target velocity, as a function of heading parameter, as a function of search [scanning] speed and as a function of the structural parameters of the station, let us turn to the analytical expression for the probability of target detection.

Let us examine the interval of time t from the instant t = 0, when

$$x = x_{max}.$$

Let us divide this time by n intervals Δt_i in each of which the probability of target detection may be presented as

$$P_i = K_i F(t_i) \Delta t_i, \tag{9}$$

where $F(t_i)$ is the average excess of useful signal over the mean noise level during the time Δt_i;
K_i is the proportionality factor.

The events – the appearance of the target at intervals Δt_i – are assumed to be independent. In this case the total probability of target detection during the time t is equal to

$$P = 1 - \exp\left[-\sum_{i=1}^{n} K_i F(t_i) \Delta t_i\right]. \tag{10}$$

Passing to the limit with $\Delta t \to 0$, we obtain

$$P = 1 - \exp\left[-\int_{t=0}^{t} \mathrm{K}F(t)\,dt\right], \tag{11}$$

where $\mathrm{K}F(t)$ is a function of range $f(x)$.

Assuming that the time t corresponds to the horizontal range x, and with $t = 0$, $x = x_{max}$, we obtain

$$P(x) = 1 - \exp\left[-\int_{x_{max}}^{x} f(x)\,dx\right]. \tag{12}$$

Having denoted $\varphi(x) = \int_{x_{max}}^{x} f(x)\,dx$, we obtain

$$P(x) = 1 - e^{-\varphi(x)}. \tag{13}$$

From this formula, cited in [50], we can calculate the increasing probability of detection as a function of horizontal range to the target with a heading parameter close to zero, if $f(x)$ and x_{max} are known.

The values of x_{max} are calculated according to Formulas (6) or (8), and $f(x)$ are determined by means of experimental data with respect to the frequency of detection for a target flying past with small heading parameters.

According to experimental data the function $f(x)$ is generally written in the form of the linear relationship

$$f(x) = \frac{x_{max} - x}{a^2}, \tag{14}$$

$$a = \chi V_{\text{ц}}(1 - e^{-\bar{\tau}}), \tag{15}$$

where V_{ts} is the target velocity;

$\bar{\tau}$ is the average time of the scanning cycle;

χ is a factor determined experimentally.

Having substituted (14) into (12), and then into (13), we obtain

$$P(x) = 1 - \exp\left[-\frac{(x_{max} - x)^2}{2a^2}\right]. \tag{16}$$

Calculation with Formula (16) permits derivation of detection probability for low-flying target as function of horizontal range x for target heading parameter close to zero if x_{max} is defined by Eqs. (6) and (8) and χ is known.

Under actual conditions a target moves directly at the unit very infrequently. Most frequently the heading of a nonmaneuvering target is a straight line passing at a random distance from the

radar station. The detection range will be a weak function of the target irradiation direction, since the reflected signal remains virtually unchanged. The power of the signal reflected from the aircraft at a wavelength of λ = 10 cm with irradiation from the front varies within limits of 15-25 db, while with a change in the flight target aspect with respect to the line of sight the power of the reflected signal varies within limits of 15-30 db ([70], page 37).

This permits the assumption that in motion of a target with a parameter different from zero, the balancing factor χ will, for all intents and purposes, be a weak function of the target heading parameter.

In this case, if we employ the approximate function $f(x)$, for the heading of a target x with the parameter z at altitude H, $\bar{\varphi}(x, H, z)$ is determined from the formula

$$\bar{\varphi}(x, H, z) = \int_{x_{max}}^{x} \frac{Д_{max} - \sqrt{H^2 + z^2 + x^2}}{a^2} dx, \tag{17}$$

where $Д_{max}$ is the maximum slant range determined from Formula (6) or (8).

The probability of target detection as a function of the horizontal range x can be calculated according to the equation

$$P(x) = 1 - e^{-\bar{\varphi}(x, H, z)}, \tag{18}$$

where H = const, z = const.

Determination of the experimental balancing factor χ will be demonstrated in an example.

EXAMPLE 2. We have carried out 200 measurements of detection range for a target flying at an altitude of 1000 m by a weapons guidance station (τ = 4 sec, x_{max} = 28 km, z = 0, v_{ts} = 400 m/ /sec). The measurement results have been reduced to a statistical series.

TABLE 1.6.2

$Д_i$, км	6; 8	8; 10	10; 12	12; 14	14; 16	16; 18	18; 20	20; 22	22; 24	24; 26	26; 28
m_i	1	3	5	17	32	37	34	30	20	13	4
P^*_i	0,01	0,01	0,05	0,08	0,16	0,19	0,17	0,15	0,10	0,06	0,02

Determine the detection frequency $F^*(Д)$ of the target and the balancing factors a and χ of the theoretical and experimental values of the detection frequencies.

Solution. Having defined the detection frequency as the distribution function $F^*(Д)$ (Table 1.6.3), from Eq. (16) we find a = 7.7 km when $F^*(Д)$ = 0.5, while from Eq. (15) we find χ = 19.5.

The coefficient χ makes it possible to determine the target detection probability theoretically as a function of scanning conditions.

TABLE 1.6.3

$Д_i$, км	7	9	11	13	15	17	19	21	23	25	27
$F^*(\bar{Д})$	1,00	0,99	0,98	0,93	0,85	0,69	0,50	0,33	0,18	0,08	0,02
$P(\bar{Д})$	0,99	0,94	0,92	0,86	0,76	0,65	0,50	0,34	0,19	0,08	0,01

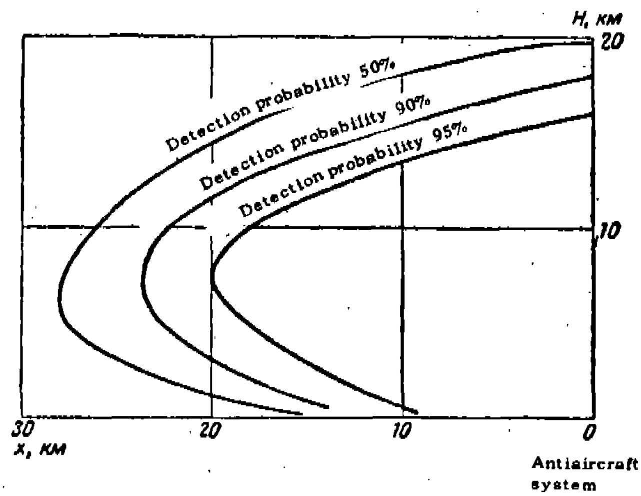

Fig. 1.6.1

Table 1.6.3 gives the target detection frequency $F^*(\bar{Д})$ determined experimentally, and also the detection probability $P(\bar{Д})$ derived from Formula (16). Instead of $Д_i$ we assumed x_i which in this case are virtually identical. We can see from Table 1.6.3 that the coincidence of the theoretical and experimental values for detection probability is quite satisfactory.

To evaluate efficiency of an antiaircraft system as a function of horizontal range x and altitude H according to Eq. (18) we calculate the target detection probability. Isoprobable curves (Fig. 1.6.1) are then plotted and the efficiency of the system is calculated.

§1.7. COMPONENT [ELEMENT] RELIABILITY CRITERIA

A. General Statements

Since the elements are irreplaceable items (see §0.7), the reliability characteristic of each specific element will be its individual service life t calculated from the start of element service to the instant of element breakdown. Here, of course, we have in mind certain specific operational conditions (climatic conditions, load conditions, vibrations and acceleration, etc.).

Examination of numerous similar elements reveals their service lives to be random quantities. Let $f(t)$ denote the distribution density of the element's service-life duration. We can then present the following equations for the basic element reliability criteria (see [82], page 364).

1. *Probability of faultless element operation* during the period of time t

$$P(t)=\int_{t}^{\infty} f(t)\,dt. \tag{1}$$

2. *Average element service life*

$$t_{\text{cp}}=\int_{0}^{\infty} t f(t)\,dt=\int_{0}^{\infty} P(t)\,dt. \tag{2}$$

3. *Element failure rate*

$$\lambda(t)=\frac{f(t)}{P(t)}=-\frac{1}{P(t)}\frac{dP(t)}{dt}. \tag{3}$$

The physical significance of this criterion will be clarified below.

Let us stress the following circumstance. Knowing any of the three functions $f(t)$, $p(t)$ and $\lambda(t)$, we can determine the remaining two. Indeed, having integrated Eq. (3), we obtain

$$P(t)=\exp\left(-\int_{0}^{t}\lambda(t)\,dt\right). \tag{4}$$

Equations (1), (3) and (4) make it possible, from any of the indicated three functions, to find the remaining two.

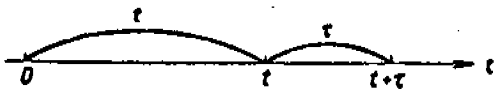

Fig. 1.7.1

Let us examine two successive time intervals t and τ (Fig. 1.7.1). The probability of faultfree element operation in the intervals from 0 to t and from 0 to $t + \tau$ will correspondingly be equal to $P(t)$ and $P(t + \tau)$. Let $P_t(\tau)$ denote the conditional probability of faultfree element operation in the interval from t to $t + \tau$, calculated for the condition that at the instant t the element was repaired. According to the probability multiplication theorem we can write

$$P(t+\tau)=P(t)P_t(\tau),$$

hence

$$P_t(\tau)=\frac{P(t+\tau)}{P(t)}. \qquad (5)$$

From Eqs. (4) and (5) we obtain

$$P_t(\tau)=\exp\left(-\int_t^{t+\tau}\lambda(t)\,dt\right)=e^{-\tau\lambda_{cp}}, \qquad (6)$$

where

$$\lambda_{cp}=\frac{1}{\tau}\int_t^{t+\tau}\lambda(t)\,dt. \qquad (7)$$

To ascertain the physical significance of the failure rate let us examine Eq. (6) for the special case in which the interval τ is very small, and namely $\tau = \Delta t$. In this case, obviously, $\lambda_{sr} = \lambda(t)$ and from Eq. (6) we find

$$P_t(\Delta t)=e^{-\Delta t\lambda(t)}\cong 1-\Delta t\lambda(t). \qquad (8)$$

Hence we derive the failure probability for the time interval from the instant t to the instant $t + \Delta t$ for the condition that at the instant of time t the element was repaired

$$q_t(\Delta t)\cong \Delta t\lambda(t). \qquad (9)$$

It follows from this equation that the failure rate at the given instant of time t is equal to the failure probability per unit time close to that instant t (under the condition that at the instant t the element was repaired).

Experience demonstrates that the failure rate for the elements frequently depends on time as shown in Fig. 1.7.2. We can see from the figure that the life of the element involves three separate periods:

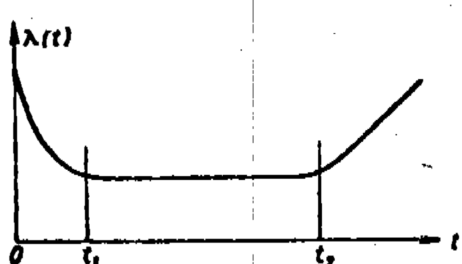

Fig. 1.7.2

1) The period from the start of operation to the instant t_1 which is referred to as the *adjustment period*, or the instant mortality period (or the period of defective-element burnout). This period is described by an elevated failure rate which is explained by the presence of hidden production defects which generally appear during the initial period of element operation.

2) The period from the instant t_1 to the instant t_2 which is characterized by a constant failure rate.

This period is referred to as the *period of normal operation.*

3) The period after the instant t_2 which is characterized by an increased rate of failure because of element aging (wear). This

period is known as the *element aging (wear)*.

It follows from Fig. 1.7.2 that one way of increasing element reliability is the so-called "training" of the elements, which involves the following. Prior to use of the elements they are held under load for a period of time t_1. During this time a portion of the elements will break down because of hidden defects or weak spots, with the remaining elements exhibiting greater reliability than the initial over-all group, since the failure rate will be lower.

B. The Exponential Service Duration Distribution Function

The service duration distribution density in the subject case is written in the form

$$f(t) = \lambda e^{-\lambda t}, \tag{10}$$

where λ is the distribution function parameter (Fig. 1.7.3).

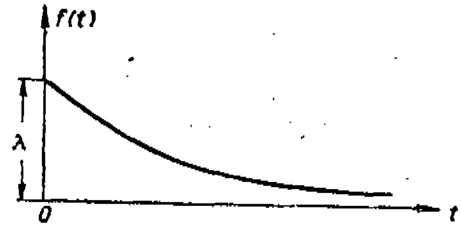

Fig. 1.7.3

From Eqs. (1)-(3) we obtain

$$P(t) = e^{-\lambda t}, \tag{11}$$

$$t_{cp} = \frac{1}{\lambda}, \tag{12}$$

$$\lambda(t) = \lambda, \tag{13}$$

i.e., the failure rate is constant here and coincides with the distribution function parameter.

The exponential function is encountered in actual practice when the elements are used on completion of the adjustment period, and aging (wear) phenomena are negligibly small. This occurs, for example, with many elements of radioelectronic apparatus: capacitors, resistors, semiconductor devices, etc. Electron tubes and shf [СВЧ] devices generally exhibit no exponential function, since wear (aging) phenomena are of great significance in the operation of these devices.

From Eq. (5) we find

$$P_t(\tau) = e^{-\lambda \tau}, \tag{14}$$

i.e., the probability of faultfree element operation during the

interval τ under the condition that it was repaired at the beginning of this interval is independent of the duration t of the previous operation. This remarkable property of the exponential distribution is explained by the absence of aging (wear) on the part of the elements in their operation. In this case, instead of the curve (Fig. 1.7.2) we have a straight line parallel to the axis of abscissas.

Hence it follows that with exponential distribution of element service duration, "training" of elements is inexpedient.

C. Service Duration Distribution According to the Weibull Function

The service duration distribution density in this case is written in the form

$$f(t)=\frac{m}{t_0}t^{m-1}\exp\left(-\frac{t^m}{t_0}\right), \tag{15}$$

where t_0 and m are the distribution function parameters (Fig. 1.7.4).

From Eqs. (1)-(3) we obtain

$$P(t)=\exp\left(-\frac{t^m}{t_0}\right), \tag{16}$$

$$t_{cp}=t_0^{\frac{1}{m}}\Gamma\left(\frac{1}{m}+1\right), \tag{17}$$

$$\lambda(t)=\frac{m}{t_0}t^{m-1}. \tag{18}$$

It follows from Eq. (18) that when $m < 1$ the failure rate diminishes with time, while when $m > 1$ the rate of failure increases with time. When $m = 1$ the Weibull function degenerates into an exponential function (Fig. 1.7.5). Thus, assuming various magnitudes for m, we can describe the entire curve in Fig. 1.7.2 by parts with the Weibull function.

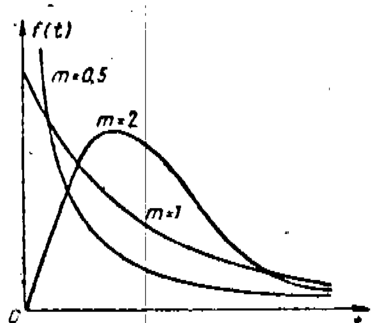

Fig. 1.7.4

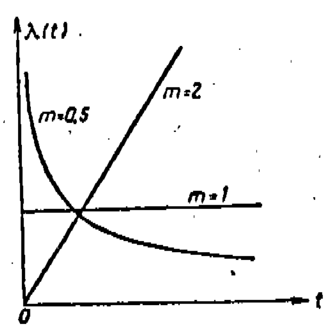

Fig. 1.7.5

Experience shows that the Weibull function with the parameter $m > 1$ is found for many types of electron tubes, shf devices, ball bearings, etc.

From Eq. (5) we obtain

$$P_t(\tau)=\exp\left(-\frac{(t+\tau)^m-t^m}{t_0}\right). \qquad (19)$$

As an example, let us consider the case in which $t_0 = 1$, $\tau = 1$, and $m = 0.5$ or $m = 1.5$. Results of calculation with Eq. (19) under these conditions are shown in Table 1.7.1.

TABLE 1.7.1

t	0	1	2	3	4	5
$P_t(\tau)$ for $m=0{,}5$	0,37	0,66	0,73	0,77	0,79	0,81
$P_t(\tau)$ for $m=1{,}5$	0,37	0,16	0,09	0,06	0,04	0,03

We can see from this table that preliminary "training" of elements when $m = 0.5$ raises the probability of their subsequent faultfree operation (adjustment occurs). When $m = 1.5$, "training" is inadvisable, since probability of subsequent faultfree operation diminishes (element wear occurs during the operating time).

D. Experimental Determination of Failure Rate

For simplicity, let us examine the case in which the service duration distribution function for the elements is exponential. The experiment in this case is formulated in the following manner. Under the given conditions n elements of the given type are tested on a stand for a period of time t. As a result of the experiment the number m of failing elements is established.

The experimental rate of failure is determined from the results of this test by the equation

$$\lambda_{\text{оп}}=\frac{m}{nt}. \qquad (20)$$

Reliable boundaries for the assumed reliable probability α are found from the equations

$$\lambda_{\text{н}}=\frac{\lambda_{\text{оп}}}{r_1}, \qquad (21)$$

$$\lambda_{\text{в}}=\frac{\lambda_{\text{оп}}}{r_2}, \qquad (22)$$

where the coefficients r_1 and r_2 are determined from Table 7 of the appendix for the given α and the experimentally derived m (see [82], page 388).

If the test reveals no failures ($m = 0$), the reliability boundaries are found from the equations

$$\lambda_{\text{н}}=0, \quad \lambda_{\text{в}}=\frac{r_0}{nt}, \tag{23}$$

where r_0 is determined from Table 7 of the appendix.

EXAMPLE 1.7.1. In testing 10,000 resistors for 1000 hours, there were 3 failures. Find the rate of failure of the resistors.

From Eq. (20) we find

$$\lambda_{\text{он}}=3\cdot 10^{-7}.$$

Assuming α = 0.95, for m = 3 from Table 7 of the appendix we find r_1 = 3.66 and r_2 = 0.39. From Eqs. (21) and (22) we find

$$\lambda_{\text{н}}=0{,}82\cdot 10^{-7}, \ \lambda_{\text{в}}=7{,}7\cdot 10^{-7}.$$

This example shows that even a large number of tests yields a rate of failure with poor relative accuracy.

§1.8. RELIABILITY CRITERIA OF IRREPLACEABLE ITEMS

A. General Statements

The reliability characteristic of each specific item in an examination of the operation of similar irreplaceable items under certain set operational conditions will be the individual service duration t of that item, reckoned from the start of service for the item to the instant that it breaks down. The situation is the same in the case of elements (see §1.7). All element reliability characteristics considered in §1.7 are therefore applicable to any irreplaceable products.

If an irreplaceable product for which there are no reserves consist of k elements exhibiting a failure rate

$$\lambda_1(t), \lambda_2(t), \ldots, \lambda_k(t),$$

the rate of failure of the product on the whole is found from the equation

$$\lambda(t)=\sum_{i=1}^{k}\lambda_i(t). \tag{1}$$

In the special case in which all elements exhibit constant rates of failure $\lambda_1, \lambda_2, \ldots, \lambda_k$, the service duration of the product will be distributed exponentially. The mean service duration t_{sr} for the entire product (to the first failure of any of the elements) in this case will be determined from the equation

$$\frac{1}{t_{\text{ср}}}=\sum_{i=1}^{k}\frac{1}{t_{\text{ср}\,i}}, \tag{2}$$

where $t_{\text{ср}\,i}=\frac{1}{\lambda_i}$ is the mean service duration of the ith element.

Equation (2) is a simple consequence of Eqs. (1) and (1.7.12).

Irreplaceable products are frequently intended for work during a predetermined interval of time t_r. Thus, for example, the radio fuze of a missile is intended for operation throughout the flight time of the missile. For a fuze t_r must therefore be equal to the flight time.

An important reliability characteristic of such products is the probability of faultfree operation during the time t_r

$$P(t_{\text{р}}) = e^{-\lambda t_{\text{р}}} = e^{-t_{\text{р}}/t_{\text{ср}}}. \tag{3}$$

The failure probability during the time t_r is also employed frequently

$$q(t_{\text{р}}) = 1 - P(t_{\text{р}}). \tag{3a}$$

When t_r is small in comparison to t_{sr}, Eq. (3a) may be written approximately in the following form:

$$q(t_{\text{р}}) = \frac{t_{\text{р}}}{t_{\text{ср}}}. \tag{4}$$

If the product functions in cycles of duration t_r, the probability of faultfree product operation in n cycles is written in the following form:

$$P(n) = e^{-\lambda n t_{\text{р}}} = (e^{-\lambda t_{\text{р}}})^n = P_1^n, \tag{5}$$

where P_1 is the probability of faultfree operation during a single cycle.

B. Reliability Characteristics in the Process of Preparation for Work

To prepare irreplaceable products for application we require a certain normal preparation time t_n. If failure of certain elements included in the product is noted during the preparation, the actual preparation time t_f will be larger than t_n due to the time spent on the determination and elimination of the faults.

Let us introduce the denotation

$$\tau = t_{\text{ф}} - t_{\text{н}}. \tag{6}$$

The preparation lag τ is random. Processing of experimental statistics will yield its distribution function. We denote its conditional probability density $\varphi(\tau)$ (given that there is a preparation lag).

In first approximation we can assume the exponential expression

$$\varphi(\tau) = \frac{1}{\tau_{cp}} \exp\left(-\frac{\tau}{\tau_{cp}}\right), \qquad (7)$$

where τ_{sr} is the mean preparation lag (for those cases in which there is preparation lag).

The conditional probability $q_y(t)$ that the preparation lag does not exceed the given time t is found from the equation

$$q_y(t) = \int_0^t \varphi(\tau)\, d\tau = 1 - \exp\left(-\frac{t}{\tau_{cp}}\right). \qquad (8)$$

Unconditional probability $q(t)$ that the preparation lag does not exceed the given time t is found from the equation

$$q(t) = 1 - (1 - P_n) \exp\left(-\frac{t}{\tau_{cp}}\right), \qquad (9)$$

where P_p is the normal preparation probability (i.e., preparation without lag).

Since preparation lag is equal to zero with probability P_p, and with probability $1 - P_p$ the mean preparation time is equal to τ_{sr}, the unconditional mean preparation lag will be

$$T_{cp} = P_n \cdot 0 + (1 - P_n)\tau_{cp} = (1 - P_n)\tau_{cp}. \qquad (10)$$

EXAMPLE 1. Let the experimentally determined values be $P_p = 0.90$ and $\tau_{sr} = 2$ hr. Find the mean preparation characteristics for 1000 products.

Solution. Since $P_p = 0.90$, $1000 \cdot 0.90 = 900$ products, on the average, will be prepared for operation without lag. On the average, 100 products will be prepared with lag. Of these 100 products, the average preparation lag time is 2 hr. The average preparation lag time for all 1000 products is found from Eq. (10)

$$T_{cp} = (1 - 0{,}9) \cdot 2 = 0{,}2 \text{ hr.}$$

Let us assume the preparation lag $t = 1$ hr. From Eq. (9) we find

$$q(t) = 1 - (1 - 0{,}9) \exp\left(-\frac{1}{2}\right) = 0{,}939.$$

This means that, on the average, of 1000 products $1000 \cdot 0.939 = 939$ products will be prepared with a lag no larger than 1 hr.

C. The Reserve Case

Let us consider the case in which a product consists of k identical independent simultaneously operating blocks. The product

is held to function faultlessly over the time interval from 0 to t if at least a single block functions faultlessly during this interval.

Let one block exhibit the rate of failure $\lambda_1(t)$ and a probability $P_1(t)$ of faultless operation during the time from 0 to t. For the entire product we will then have

$$P_k(t) = 1 - [1 - P_1(t)]^k. \tag{11}$$

The failure rate for the entire product is found from Eqs. (11) and (1.7.3)

$$\lambda_k(t) = \frac{k[1 - P_1(t)]^{k-1}\lambda_1(t)\,P_1(t)}{1 - [1 - P_1(t)]^k}. \tag{12}$$

For simplicity, let us examine the case of $k = 2$,

$$\lambda_1(t) = \text{const} = \lambda_1,\ P_1(t) = e^{-\lambda t}.$$

Here, Eqs. (11) and (12) are rewritten in the following form:

$$P_2(t) = 2e^{-\lambda t} - e^{-2\lambda t}, \tag{13}$$

$$\lambda_2(t) = \frac{2\lambda(1 - e^{-\lambda t})}{2 - e^{-\lambda t}}. \tag{14}$$

Figures 1.8.1 and 1.8.2 show the curves $P_2(t)$, $\lambda_1(t)$ and $\lambda_2(t)$ for the case $\lambda = 1$. We see from these figures that reserves yield the following results:

1. Probability of faultfree operation increases noticeably. When $k > 2$ this phenomenon is intensified.

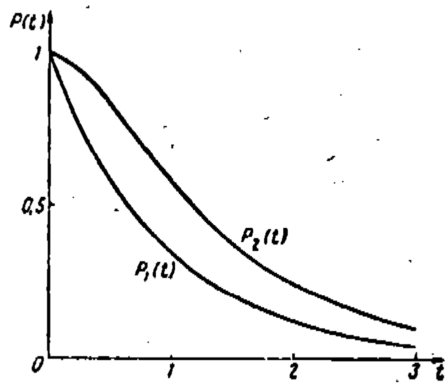

Fig. 1.8.1

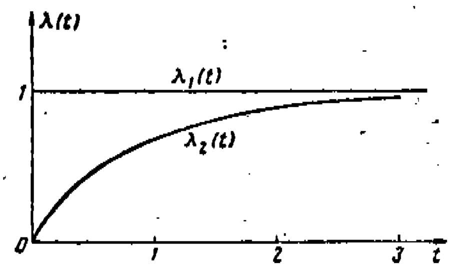

Fig. 1.8.2

2. The failure rate diminishes sharply with small t, and then asymptotically approaches the value which it exhibits in the absence of reserves.

D. Element Reserves with Disconnection and Shortcircuit Type Failures

Let us consider the case of the so-called constant element reserve in which the basic and reserve elements function simultaneously, i.e., all elements are equal.

The elements may be connected in three ways:

1. Series connection (Fig. 1.8.3).
2. Parallel connection (Fig. 1.8.4).
3. Mixed connection (Figs. 1.8.5 and 1.8.6).

Fig. 1.8.3

The element group exhibiting one of the above-indicated connections will be referred to, in the interest of brevity, as an element system.

We will distinguish two types of failure: a) disconnection and b) shortcircuiting.

With series connection of elements the "disconnection" type of failure in any one of the elements causes the failure of the element system, while the "circuit" failure type leads to the failure of the system only if it occurs in all elements of the system.

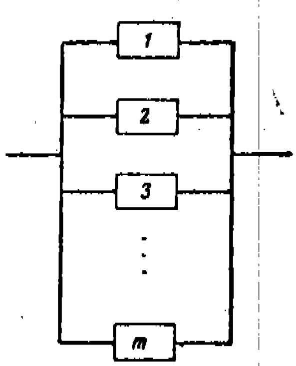

Fig. 1.8.4

With parallel element connection the "shortcircuit" failure type in only one of the elements leads to the failure of the element system, while the "disconnection" failure type leads to the failure of the element system only if it occurs in all of the elements of the system.

In the mixed element connection any type of failure in one of the elements will not cause the failure of the entire element system. Here failure of the element system may occur only with m disconnections (in all m parallel lines) or with n shortcircuits (in one of these lines).

Let us note that the subject reserve method is possible only when the parameters of the element system do not exceed the established tolerance limits for the element in reserve.

We denote by q_0 the probability of element disconnection,

and q_3 the probability of shortcircuit. Then

$$q = q_0 + q_3 \tag{15}$$

will be the probability of element failure for any reason.

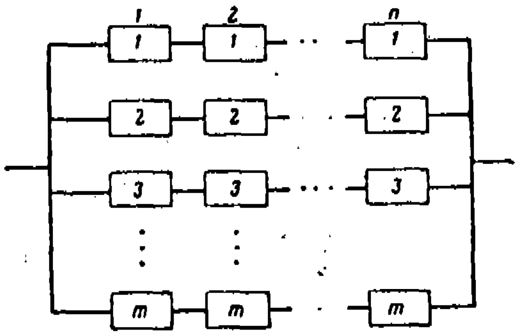

Fig. 1.8.5

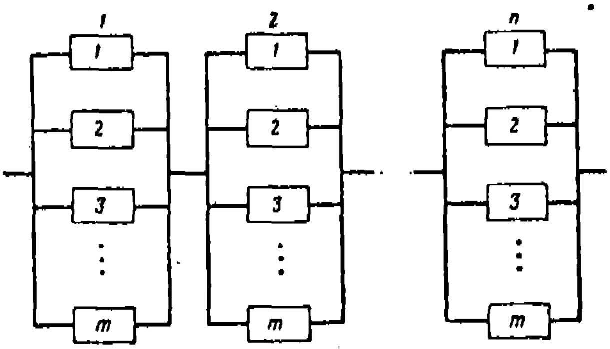

Fig. 1.8.6

We begin with the case of series element connection. Disconnection of the system will occur on disconnection in only one of the system elements. This probability will be

$$Q_{n_0} = 1 - (1 - q_0)^n. \tag{16}$$

Shortcircuiting of the system will occur on shortcircuiting of all system elements. This probability will be

$$Q_{n_3} = q_3^n. \tag{17}$$

The probability of any system failure in series connection of n elements will be

$$Q_n = 1 - (1 - q_0)^n + q_3^n. \tag{18}$$

Analogously, for a system of m parallel connected elements we will have

$$Q_{m_0} = q_0^m, \tag{19}$$

$$Q_{m_3} = 1 - (1 - q_3)^m, \tag{20}$$

$$Q_m = q_0^m + 1 - (1 - q_3)^m. \tag{21}$$

We now examine the mixed connection of elements shown in Fig. 1.8.5. This system may be treated as a parallel connection of m elements each of which exhibits the probabilities of disconnection and shortcircuiting according to Eqs. (16) and (17). The failure probability Q_{mn} of such a system may therefore be found from Eq. (21), if in this equation we substitute q_0 and q_3 for Q_{n_0} and Q_{n_3} according to Eqs. (16) and (17). This substitution yields

$$Q_{mn} = [1 - (1 - q_0)^n]^m + 1 - (1 - q_3^n)^m. \tag{22}$$

We now examine the mixed connection of elements shown in Fig. 1.8.6. This system may be regarded as the series connection of n elements each of which exhibits the probabilities of disconnection and shortcircuiting described by Eqs. (19) and (20). The failure probability Q_{nm} of such a system can be found with Eq. (18), if in this equation we substitute q_0 and q_3 with Q_{m_0} and Q_{m_3} according to Eqs. (19) and (20). This substitution yields

$$Q_{nm} = 1 - (1 - q_0^m)^n + [1 - (1 - q_3)^m]^n. \tag{23}$$

Equations (18), (21), (22) and (23) permit solution of all problems associated with the subject reserve cases.

We consider the practical important case of sufficiently small probabilities q_0 and q_3 (each less than 0.01). Equations (18), (21), (22) and (23) can then be simplified if we limit ourselves to the first two terms in an expansion in a Newton binomial. This yields approximate equations whose accuracy is fully adequate for practical purposes

$$Q_n = nq_0 + q_3^n. \tag{24}$$

$$Q_m = q_0^m + mq_3, \tag{25}$$

$$Q_{mn} = (nq_0)^m + mq_3^n, \tag{26}$$

$$Q_{nm} = nq_0^m + (mq_3)^n. \tag{27}$$

We introduce the denotation

$$k = \frac{q_0}{q}, \tag{28}$$

where q is defined by Eq. (15).

We then have

$$q_0 = kq, \tag{29}$$

$$q_3 = (1 - k)q. \tag{30}$$

With these equations we can rewrite Eqs. (24)-(27) to the form

$$Q_n = nkq + (1-k)^n q^n, \qquad (31)$$

$$Q_m = k^m q^m + m(1-k)q, \qquad (32)$$

$$Q_{mn} = (nk)^m q^m + m(1-k)^n q^n, \qquad (33)$$

$$Q_{nm} = nk^m q^m + m^n (1-k)^n q^n. \qquad (34)$$

We see from Eq. (31) that when $nk > 1$, $Q_n > q$, i.e., series connection of elements with $nk > 1$ is inexpedient, since failure probability with this connection increases. Analogously, from Eq. (32) we have that when $m(1 - k) > 1$, $Q_m > q$. Hence, parallel connection is inexpedient when $m(1 - k) > 1$.

As an example, we consider the case of a comparison of Eqs. (31)-(34) for an identical number of elements in the system. We assume the number to be equal to 4. Equations (31)-(34) are then written in the form

$$Q_n = 4kq + (1-k)^4 q^4, \qquad (35)$$

$$Q_m = k^4 q^4 + 4(1-k)q, \qquad (36)$$

$$Q_{mn} = 4k^2 q^2 + 2(1-k)^2 q^2, \qquad (37)$$

$$Q_{nm} = 2k^2 q^2 + 4(1-k)^2 q^2. \qquad (38)$$

TABLE 1.8.1

k	$q = 0,01$				$q = 0,0001$			
	q/Q_n	q/Q_m	q/Q_{mn}	q/Q_{nm}	q/Q_n	q/Q_m	q/Q_{mn}	q/Q_{nm}
0	10^6	0,25	50	25	10^{12}	0,25	5000	2500
0,001	250	0.25	50	25	250	0,25	5000	2500
0,1	2,5	0,28	60	31	2,5	0,28	6000	3100
0,5	0,5	0,5	67	67	0,5	0,5	6700	6700
0,9	0,28	2,5	31	60	0,28	2,5	3100	6000
0.999	0,25	250	25	50	0,25	250	2500	5000
1	0,25	10^6	25	50	0,25	10^{12}	2500	5000

For q let us assume the values 0.01 and 0.0001. Table 1.8.1 shows results of calculations by these equations for various k. This table shows the q/Q ratios which indicate the reliability gain resulting from the reserves (if these ratios are larger than 1) or the inadvisability of reserves (when these ratios < 1).

This table shows that reliability is increased with mixed connection for any k. The reliability gain increases with a reduction in q. With series element connection reliability increases only with rather small k, while with parallel connection, reliability increases only with rather large k.

E. Experimental Reliability Characteristic Determination

The failure rate for irreplaceable products is determined experimentally in complete analogy with that for elements (see §1.7.D).

We consider the problem of experimental determination of faultless operating probability within a specified time. The

experiment is set up as follows. We test n products under operational load for time t_r and establish the number m of products which failed. The experimental failure frequency is found from the equation

$$q_{\text{оп}} = \frac{m}{n}, \tag{39}$$

and the reliable boundaries for reliable probability $\alpha = 0.95$ is found from the equations

$$q_{\text{н}} = \frac{q_{\text{оп}}}{R_1}, \tag{40}$$

$$q_{\text{в}} = \frac{q_{\text{оп}}}{R_2}, \tag{41}$$

where the coefficients R_1 and R_2 are found from Table 6 of the appendix according to the values of m and m/n (see [82], page 193).

If the experiment yielded no failures ($m = 0$), the reliable boundaries are found from the equations

$$q_{\text{н}} = 0, \quad q_{\text{в}} = \frac{R_0}{n}, \tag{42}$$

where R_0 is determined from Table 6 of the appendix from the given reliable probability α and the quantity n.

The experimental probability of faultless operation and the reliable boundaries for that probability are determined from the equations

$$P_{\text{оп}} = 1 - q_{\text{оп}}, \tag{43}$$

$$P_{\text{н}} = 1 - q_{\text{в}}, \tag{44}$$

$$P_{\text{в}} = 1 - q_{\text{н}}. \tag{45}$$

We note that when $(m/n) < 0.10$ in Eqs. (40)-(42) we can replace the coefficients R_1, R_2 and R_0 by the coefficients r_1, r_2 and r_0 defined from Table 7 of the appendix.

EXAMPLE 2. In testing $n = 1000$ products we obtained $m = 100$ failures. Evaluate the probability of faultfree operation.

Solution. Here $q_{\text{оп}} = \frac{m}{n} = 0{,}10$. From Table 6 of the appendix for $m = 100$ and $(m/n) = 0.10$ we find $R_1 = 1.18$ and $R_2 = 0.86$. From Eqs. (40) and (41) we find

$$q_{\text{н}} = \frac{0{,}10}{1{,}18} = 0{,}085, \quad q_{\text{в}} = \frac{0{,}10}{0{,}86} = 0.116.$$

From Eqs. (43)-(45) we find $P_{op} = 0.90$, $P_n = 0.884$, $P_v = 0.915$.

EXAMPLE 3. In testing 100 fuzes we found not a single failure. Evaluate the probability of faultless operation.

From Table 6 of the appendix for $n = 100$ and $\alpha = 0.95$ we find $R_0 = 2.95$. From Eqs. (42) and (44) we find $q_v = (2.95/100) = 0.03$, $P_n = 0.97$. Obviously, $P_v = 1$.

§1.9. RELIABILITY CRITERIA FOR REPLACEABLE ITEMS

A. Nonreserve Cases

The work of replaceable items generally begins on connection of apparatus. One of the reliability criteria should therefore be the probability P_v of normal connection.

In well adjusted devices the quantity P_v is generally very close to 1. However, in tests and adjustment this quantity may differ significantly from unity. Apparatus failures in this case, on connection, should be counted separately from failures which arise during operation (see [82], page 452).

Further, for the reliability characteristics of replaceable products we should consider the parameters of the so-called period of product adjustment. In this period which marks the beginning of operation for newly fabricated items, the element and installation failure rate is elevated. The basic parameters of this period, adequate for practical purposes, are the average duration t_{pr} of this period and the average number m_{pr} of failures during this period.

On completion of the adjustment period it is generally possible to treat the occurrence of failures in product operation as a simple flow (see §4.1). The basic failure flow characteristic is the flow parameter Λ

$$\Lambda = \frac{1}{T}, \tag{1}$$

where T is the average time to failure or the average time of faultless operation.

With the parameter Λ the probability of product faultfree operation probability for any time interval t is easily found:

$$P(t) = e^{-\Lambda t} = e^{-t/T}. \tag{2}$$

If the apparatus is switched on at the start of the time t interval, the probability of faultfree operation during the time t will be

$$P_v(t) = P_v e^{-t/T}, \tag{3}$$

where P_v is the normal connection probability.

The basic characteristics of suitability for repair in replaceable products is the average restoration time T_v and the readiness factor

$$K_r = \frac{T}{T + T_v}. \tag{4}$$

The quantity T_v defines the average product idle time due to efforts to seek out and eliminate failures.

The readiness factor represents the probability of finding the subject product in operating condition under the condition that it is examined over a sufficiently long period of operating time.

In Eq. (4) we consider only the down time resulting from product repair after failure determination. At the same time, down time occurs in preventive maintenance as well. This down time may be taken into consideration by means of the use factor

$$K_{и}=\frac{t_{раб}}{t_{раб}+t_{рем}+t_{проф}}, \qquad (5)$$

where t_{rab} is the duration of proper product operation over a sufficiently large interval of time t;
t_{rem} is the down time due to measures to eliminate failures found during time t;
t_{prof} is the down time due to preventive maintenance during time t.

Down time due to other causes in this case (for example, service personnel vacations) is not considered. In the subject period t let there have been m failures in the time t_{rab}. Dividing the numerator and the denominator of Eq. (5) by m, we obtain

$$K_{и}=\frac{T}{T+T_{в}+T_{проф}}, \qquad (6)$$

where T_{prof} is the average preventive maintenance time per single failure occurring in time t_{rab}. We note here that the number m does not include failures ascertained during the preventive maintenance work. We also note that the time spent on elimination of failures determined during the preventive maintenance is not included in t_{rem} but in t_{prof}.

B. Reserve Cases Without Replenishment of Reserves

We now consider reserve cases in which reserve elements (blocks) breaking down in operation are not replenished and on which repairs are carried out only after the failure of the basic and all reserve elements (i.e., on breakdown of the entire product). This case occurs when the inoperative product is sent for repair to special repair subsections.

We consider reserve substitution cases in which there is a single basic element and $n - 1$ reserve elements. On breakdown of the basic element it is replaced by one of the reserve elements. Failure of a system consisting of n subject elements occurs when the last of the reserve elements breaks down.

We present the equations required for two reserve variants.

1) Loaded (hot) reserve in which the basic and all reserve

elements are in a single operating regime (under load).

2) Unloaded (cold) reserve in which the basic element is in the operating regime, and all reserve elements are awaiting inclusion in the operating regime (not under load).

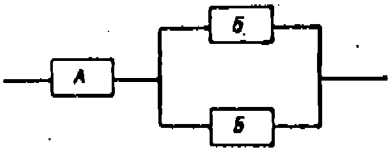

Fig. 1.9.1

For the sake of simplicity we will consider the case in which all subject elements (blocks) have passed through the adjustment (training) period and for each of which Eq. (2) is valid.

With loaded reserve the probability of faultless operation of a system of n elements (blocks) during time t is written in the form (see [82]):

$$P_n(t) = 1 - \left[1 - \exp\left(-\frac{t}{T}\right)\right]^n. \qquad (7)$$

The average time of faultless operation for a system of n elements will be

$$T_n = T\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right). \qquad (8)$$

With unloaded reserve, instead of Eqs. (7) and (8), we will have

$$P_n(t) = e^{-\frac{t}{T}} \sum_{i=0}^{n-1} \frac{t^i}{T^i i!}. \qquad (9)$$

$$T_n = nT. \qquad (10)$$

We note that Eqs. (8) and (10) are derived by integration of Expressions (7) and (9) in limits from 0 to ∞ [see Eq. (1.7.2)].

As an example, let us consider the product whose diagram is shown in Fig. 1.9.1. Here block A has no reserve and a loaded reserve is used for block Б. The probability of faultless operation for block A is found from Eq. (2)

$$P_A(t) = e^{-t/T_A}. \qquad (11)$$

The probability of faultless operation for a system of two blocks Б is found from Eq. (7)

$$P_Б(t) = 2e^{-t/T_Б} - e^{-2t/T_Б}. \qquad (12)$$

The probability of faultfree operation for the entire product is found from the theorem of probability multiplication

$$P(t) = e^{-t/T_A}\left[2e^{-t/T_Б} - e^{2t/T_Б}\right]. \qquad (13)$$

The average time T of faultfree operation for the entire product is found by integration of Expression (13) in limits from 0 to ∞ [see Eq. (7.1.2)]. On integration we obtain

$$T = \frac{T_A T_B (3T_A + T_B)}{(T_A + T_B)(2T_A + T_B)}. \qquad (14)$$

The product readiness factor is found from Eq. (4) where T is defined by Eq. (14) and T_v is the average time to repair the failure of the entire product (can be determined experimentally).

C. Reserve Cases with Reserve Restoration

For simplicity and brevity we limit ourselves to cases in which a product consists of two identical blocks of which one is operative and the other is put into operation only on appearance of failure in the first (unloaded reserve). Restoration of the broken block begins immediately on failure. We assume that the work time prior to failure and duration of restoration for each block are subject to exponential laws with rates

$$\lambda = \frac{1}{T} \quad \text{and} \quad \mu = \frac{1}{T_v}.$$

We calculate the probability $P(t)$ of faultless product operation in the time interval from 0 to t if at the instant $t = 0$ both blocks are operative. For this purpose we will seek the distribution density $f(t)$ for the product operating duration t prior to first failure.

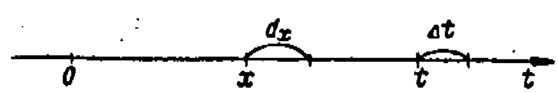

Fig. 1.9.2

If at the instant t the product fails, this means that at some instant x there occurred the failure of a single block, the restoration of this block had not been concluded at the instant t and the second block broke down at the instant t (Fig. 1.9.2).

Let the ith failure of the block occur at the instant x. We can then write

$$f(t)\,\Delta t = \sum_{i=1}^{\infty} \int_0^t f_i(x)\,dx\,P(x)\,e^{-\lambda(t-x)}\,e^{-\mu(t-x)}\lambda e^{-\lambda(t-x)}\Delta t. \qquad (15)$$

Here $f_i(x)\,dx$ is the appearance probability of ith block failure near instant x;

$P(x)$ is the probability of no product failure prior to instant x;

$e^{-\lambda(t-x)}$ is the probability of faultless block operation from instant x to instant t;

$e^{-\mu(t-x)}$ is the probability that repair of the block failing at instant x will not be completed at instant t;

$\lambda e^{-\lambda(t-x)}\Delta t$ is the probability of block failure near instant t.

Equation (15) can be rewritten to the form

$$f(t)=\lambda\int_0^\infty P(x)\,e^{-\varepsilon(t-x)}\,dx\sum_{i=1}^{\infty} f_i(x), \tag{16}$$

where

$$\varepsilon=2\lambda+\mu. \tag{17}$$

From Eqs. (100.3), (101.3) and (101.4) (see [82]) we have

$$\sum_{i=1}^{\infty} f_i(x)=\lambda. \tag{18}$$

From Eqs. (16) and (18) we obtain

$$e^{\varepsilon t}f(t)=\lambda^2\int_0^t P(t)\,e^{\varepsilon x}dx. \tag{19}$$

Differentiating both parts of Eq. (19) with respect to t and taking into consideration Eq. (1.7.1), we derive the differential equation

$$P''(t)+\varepsilon P'(t)+\lambda^2 P(t)=0. \tag{20}$$

The initial conditions here have the form

$$P(0)=1,\quad P'(0)=-f(0)=0.$$

Hence we find the solution

$$P(t)=\frac{a_1}{a_1-a_2}e^{-a_2 t}-\frac{a_2}{a_1-a_2}e^{-a_1 t}, \tag{21}$$

where

$$a_1=\frac{1}{T}(1+\gamma+\sqrt{2\gamma+\gamma^2}); \tag{22}$$

$$a_2=\frac{1}{T}(1+\gamma-\sqrt{2\gamma+\gamma^2}); \tag{23}$$

$$2\gamma=\frac{T}{T_B}=\frac{\mu}{\lambda}. \tag{24}$$

The average time T_1 of faultless operation of the entire product is found by integration of Expression (21) in limits from 0 to ∞ [see Eq. (1.7.2)]. After integration we have

$$T_{\text{п}}=2T(1+\gamma). \tag{25}$$

We note that when the reserve is not restored, according to Eq. (10) we would obtain

$$T_2=2T. \tag{26}$$

Equations (25) and (26) yield a gain which provides for restoration of the reserve

$$\frac{T_{s'}}{T_s} = 1 + \gamma. \quad (27)$$

In actual practice the quantity γ generally varies from several units to several tens. The gain from Eq. (27) is therefore quite substantial.

We consider the practical important case in which the quantity γ is large in comparison with 1. In approximate terms it may then be written

$$\sqrt{2\gamma+\gamma^2} = \sqrt{(1+\gamma)^2-1} = (1+\gamma)\sqrt{1-\frac{1}{(1+\gamma)^2}} \approx$$
$$\approx (1+\gamma)\left[1-\frac{1}{2(1+\gamma)^2}\right] = 1+\gamma-\frac{1}{2(1+\gamma)}. \quad (28)$$

For the case of large γ Eqs. (22), (23) and (28) yield

$$a_1 = \frac{1}{T}\left(2+2\gamma-\frac{1}{2+2\gamma}\right), \quad (29)$$

$$a_2 = \frac{1}{T}\cdot\frac{1}{2+2\gamma}, \quad (30)$$

$$\frac{a_1}{a_1-a_2} = \frac{2+2\gamma-\frac{1}{2+2\gamma}}{2+2\gamma-\frac{2}{2+2\gamma}} \approx 1, \quad (31)$$

$$\frac{a_2}{a_1-a_2} = \frac{\frac{1}{2+2\gamma}}{2+2\gamma-\frac{2}{2+2\gamma}} \approx 0. \quad (32)$$

From Eqs. (21) and (25) we then obtain

$$P(t) = \exp\left(-\frac{t}{T(2+2\gamma)}\right) = e^{-t/T_s}. \quad (33)$$

Thus for a restorable product with a restorable reserve the same equation as for irreplaceable products on completion of the adjustment (training) period is valid (see §1.7).

D. Experimental Determination of Reliability Characteristics

To determine the probability of normal connection we can use the method of determining the probability of faultless operation, covered in §1.8.

We now consider the problem of the experimental determination of the average time to failure. Let n products be subjected to tests, the times to failure during the period of the tests for these products given, respectively, by $t_1, t_2, \ldots, t_n$. Let $m_1, m_2, \ldots, m_n$ failures be recorded for these products in this case. The experimental average time to failure is found from the equation

$$T_{\text{оп}} = \frac{t_1+t_2+\ldots+t_n}{m_1+m_2+\ldots+m_n}. \quad (34)$$

and the reliable boundaries for the given reliable probability α is found from the equations

$$T_{н}=r_2T_{оп}, \tag{35}$$

$$T_{в}=r_1T_{оп}, \tag{36}$$

where the coefficients r_1 and r_2 are found from Table 7 of the appendix on the basis of the given probability α and the total number of failures m (see [82], page 450).

If there were no failures during the tests, the reliable boundaries are found from the equations

$$T_{н}=\frac{1}{r_0}(t_1+t_2+\ldots+t_n), \qquad T_{в}=\infty, \tag{37}$$

where the coefficient r_0 is also found from Table 7 of the appendix.

EXAMPLE 1. Three devices were tested. The test times for these devices amounted to 500, 700 and 400 hr, with the number of failures 6, 8 and 3, respectively. Evaluate the average time to failure.

Solution. From Eq. (34) we find

$$T_{оп}=\frac{500+700+400}{6+8+3}=94 \text{ hr.}$$

From Table 7 of the appendix for $\alpha = 0.95$ and $m = 17$ we find $r_1 = 1.58$ and $r_2 = 0.67$. From Eqs. (35) and (36) we determine $T_n = 63$ hr, $T_v = 149$ hr.

EXAMPLE 2. In testing apparatus for 100 hr there were no failures. Evaluate the average time to failure.

Solution. We assume the reliable probability $\alpha = 0.95$. From Table 7 of the appendix we find $r_0 = 3$. From Eq. (37) we determine $T_n = 33.3$ hr.

§1.10. ASSEMBLY RELIABILITY CRITERIA

In §1.7-1.9 we considered the reliability criteria for elements of irreplaceable and replaceable products. Any armament system consists of many products and we can say that the assembly [system] reliability is defined by the totality of reliability criteria for all of the parts included in that system.

There arises the question as to whether it would not be possible to characterize the reliability of the system as a whole with some single criterion. It develops that this is possible only in the simplest cases. The reliability of the system as a whole in these cases is characterized by the probability of its normal functioning

$$P_{к}=P_1P_2P_3\ldots P_n, \tag{1}$$

where P_k is the probability of normal (faultless) functioning of the system, while P_1, P_2, ..., P_n are the probabilities of faultless functioning of parts (elements, blocks) included in the system.

Equation (1) is a rather good characterization of system reliability in the simplest case in which the breakdown of any part of the system leads to the breakdown of the entire system. In complex cases in which the system operates with several parallel channels, stations, etc., Eq. (1) is not longer adequate as a characteristic of system reliability.

As an example, let us consider the case in which the system consists of a single launch installation and n rockets. It is evident that the breakdown of one or several rockets does not indicate the breakdown of the entire system. In this case the probability of normal functioning, calculated with Eq. (1), is inadequate to characterize the system reliability. In this case we can introduce the following as additional reliability indicators:

the probability $P_{m,n}$ of normal system functioning in the case of m launches from a total number n launches;

the mathematical expectation $M(m)$ of the number of normal launches from a total number of n launches.

We present equations for these system reliability indicators for the case in which the reliability function of the launch installation is exponential. This means that the probability of the normal functioning of the launch installation in the case of k successive launches is equal to P_1^k, where P_1 is the probability of the normal functioning of the launch installation on the first launch.

Let us introduce the denotation Q_k for the probability of the following event: the launch installation functioned normally in all launches from the first to the launch having the number k (inclusive) and failed on the $(k + 1)$th launch. Then, obviously, we have

$$
\begin{aligned}
&Q_0 = 1 - P_1,\\
&Q_1 = P_1(1 - P_1),\\
&Q_2 = P_1^2(1 - P_1),\\
&\dots\dots\dots\dots\\
&Q_{n-1} = P_1^{n-1}(1 - P_1),\\
&Q_n = P_1^n,
\end{aligned}
\qquad (2)
$$

where Q_n is the probability that the launch installation in all n launches functioned normally. It is easily proved by direct addition that the sum of all Probabilities (2) is equal to unity

$$\sum_{k=0}^{n} Q_k = 1, \tag{3}$$

as was to be expected, since we are dealing here with a total group of events.

We introduce the denotation R_1 for the probability of the normal functioning of a single rocket. We can then write the equation

$$P_{m,n} = Q_m R_1^m + Q_{m+1} C_{m+1}^m R_1^m (1-R_1) + \\ + Q_{m+2} C_{m+2}^m R_1^m (1-R_1)^2 + \ldots + \\ + Q_{n-1} C_{n-1}^m R_1^m (1-R_1)^{n-m+1} + Q_n C_n^m R_1^m (1-R_1)^{n-m}. \tag{4}$$

For brevity we introduce the denotation

$$z = P_1 (1-R_1). \tag{5}$$

From Eqs. (2), (4) and (5) we obtain

$$P_{m,n} = P_1^m R_1^m (1-P_1) \sum_{k=0}^{n-m-1} C_{m+k}^m z^k + \\ + C_n^m P_1^m R_1^m z^{n-m}. \tag{6}$$

The probability of having no less than m normal launches out of n will be

$$P^*_{m,n} = \sum_{i=m}^{n} P_{i,n}. \tag{7}$$

In the special case $m = 0$ from Eq. (6) after simple transformations we obtain

$$P_{0,n} = (1-P_1)\frac{1-z^n}{1-z} + z^n. \tag{8}$$

Hence for the probability of having at least one normal launch out of n we have

$$P^*_{1,n} = 1 - P_{0,n} = (1-z^n)\frac{P_1 - z}{1-z}. \tag{9}$$

For the mathematical expectation of the number of normal launches out of n we have (by definition of mathematical expectation)

$$M(m) = \sum_{i=1}^{n} i P_{i,n}. \tag{10}$$

We present an example. Let $n = 3$, $P_1 = R_1 = 0.9$. From Eq. (5) we find $z = 0.09$. From Eq. (8) we find $P_{0.3} = 0.11$. By means of Eq. (6) we determine $P_{1.3} = 0.10$, $P_{2.3} = 0.24$ and $P_{3.3} = 0.53$. From Eq. (9) we find $P^*_{1.3} = 0.89$.

With Eq. (10) we determine $M(m) = 2.17$. Hence we find the mean fraction of normal launches

$$\frac{M(m)}{n}=\frac{2.17}{3}=0,72.$$

We note the probability of the normal functioning in the first launch to be larger – it is equal to $P_1R_1 = 0.81$.

We have considered the simple case of a system whose reliability can be characterized by the probability indicators (6)-(10) which are functions of the reliability indicators for the component parts of the system and only of these.

With more complex systems it is impossible to characterize their reliability by means exclusively of the reliability indicators of the component parts of the system – it becomes necessary to resort additionally to combat application effectiveness indicators.

In the simplest case the combat application effectiveness of a system can be characterized by the probability $\bar{R}$ of executing the combat assignment

$$\bar{R}=P_{\kappa}R_{0}, \tag{11}$$

where R_0 is the conditional probability of executing the combat assignment under the condition that all elements of the system are functioning faultlessly. Obviously, R_0 is a function of system accuracy, effect against target and similar characteristics of system quality, but is independent of system reliability. System reliability characteristics are included in the factor P_k.

Let us now consider the case of one or more system parts breaking down without causing the breakdown of the entire system, but simply reducing its effectiveness.

As before, let the system consist of n parts for which the probability of faultless operation is equal to $P_1, P_2, \ldots, P_n$. The probability of the system completing the combat assignment under the condition of faultless operation of all of its parts is denoted R_0. On breakdown of only the single ith part of the system, let the probability of execution of the combat assignment be equal to R_i.

On breakdown of only two parts (the ith and the jth) of the system, the probability of execution of the combat assignment will be equal to R_{ij}.

For simplicity, we consider the case of breakdown of three or more system parts where the probability of combat-assignment execution becomes equal to zero. Then, for the unconditional probability $\bar{R}$ of combat-assignment execution we have the equation

$$\bar{R}=P_{\kappa}R_0+\sum_{i=1}^{n}Q_iR_i+\sum_{\substack{ij=1\\ i\neq j}}^{n}Q_{ij}R_{ij}, \tag{12}$$

where Q_i is the probability of that state of the system in which only the ith part has broken down, and

Q_{ij} is the probability of that condition of the system in which only the ith and jth parts have broken down.

With the breakdown of individual system parts independent, we can write the equations

$$Q_i = P_1 P_2 \ldots P_{i-1}(1-P_i) P_{i+1} \ldots P_n, \tag{13}$$

$$Q_{ij} = P_1 P_2 \ldots P_{i-1}(1-P_i) P_{i+1} \ldots P_{j-1}(1-P_j) P_{j+1} \ldots P_n. \tag{14}$$

We illustrate Eqs. (12)-(14) by means of an example. Let $n=3$, $P_1 = P_2 = P_3 = P$, $R_1 = 0$,

$$R_{12} = R_{13} = 0.$$

From Eqs. (12)-(14) we then find

$$\tilde{R} = R_0 P^3 + R_2 P^2 (1-P) + R_3 P^2 (1-P) + R_{23} P (1-P)^2. \tag{15}$$

We consider four versions of numerical values for the quantities in Eq. (15). These versions and the calculational results for these from Eqs. (1) and (15) are shown in Table 1.10.1.

This table shows that probability $\tilde{R}$ for execution of the combat assignment may be higher in the case of a complex system than the probability P_k for the normal functioning of the entire system. With a simple system it follows from Eq. (11) that $\tilde{R} \leq P_k$ always (since $R_0 \leq 1$).

TABLE 1.10.1

Version	R_0	R_2	R_3	R_{23}	P	$\tilde{R}$	P_k
A	0.90	0.80	0.70	0.50	0.80	0.65	0.51
B	0.90	0.50	0.30	0.10	0.85	0.64	0.61
C	0.90	0.85	0.85	0.80	0.75	0.66	0.42
D	0.70	0.50	0.30	0.10	0.85	0.52	0.61

This table also shows that the probability P_k cannot serve as a comparative evaluation of the quality of two systems. Indeed, the table shows that a system with a smaller P_k may exhibit higher effectiveness $\tilde{R}$. Thus, for example, P_k in system A is smaller than for system D, while the effectiveness of A is greater than for D.

This example shows that P_k for the normal functioning of the entire complex cannot serve as the reliability criterion for a complex system. Proper evaluation of reliability in complex systems can be achieved by examination of their effectiveness criteria during whose calculation the reliability criteria of the component parts of the system are taken into consideration.

§1.11. ARMAMENT COST CHARACTERISTICS AND THEIR DETERMINATION

A. The Concept of Armament Cost

The cost of armament is one of its most important characteristics, governing to a great extent the feasibility and possible scope of its application. Armament cost increases continuously and analysis of this factor is therefore becoming increasingly urgent. Figure 1.11.1 shows a curve taken from Reference [38] indicating the variation in the percentage of expenditures on military equipment in the over-all cost of material military expenditures. Analogous data for the USA during the Second World War are presented. The curve shows that the expenditures on armament increase both in relative and absolute terms. It is interesting that the USA expenditures for research and development in armaments shows a substantial increase.

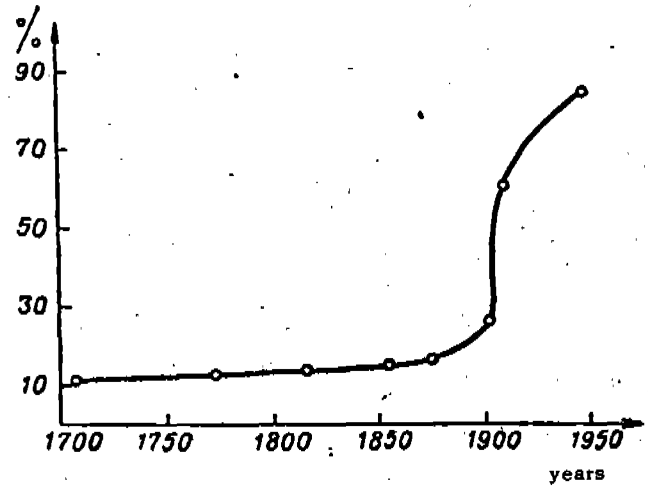

Fig. 1.11.1

In speaking of armament costs it is necessary to specify precisely what we are talking about. First of all, we must take into consideration the circumstance that armament consists not only of rockets, aircraft, military vehicles and related military units, but of an entire complex of auxiliary machinery, devices, etc. The costs of these various complexes must therefore be compared, bearing in mind that, for example, the least expensive rocket does not always correspond to the least expensive system. Speaking of the cost of a complex [system], we should always bear in mind the stage of equipment servicing from which we consider that system (factory, base, technical utilization, etc.).

Secondly, consideration should be given to the expenditure volume for research projects, armament development, armament fabrication at factories, storage, etc. Of course, in these comparisons we can limit ourselves to an analysis of individual expenditures, since these expenditures, in part, may not be the most important, nor identical; however, in all cases there should be a preliminary consideration of all expenditures in order to avoid gross errors.

For example, in resolving the problem of the introduction feasibility of a new item, even if inexpensive, we must take into consideration the expenditures on its development, since these may have been significant.

Development expenditures should be understood to refer to those costs incurred on theoretical research, on scientific and experimental-design projects, on fabrication of experimental models and on their testing. The latter may call for the development of special test areas, as well as for the development of test-area equipment. Moreover, the tests are not always successful and additional processing of the model may be required.

Thirdly, the significant effect of production volume on cost should be borne in mind: single specimens cost considerably more than those mass produced.

Finally, an important indicator characterizing the cost of an item is the number and quality of the servicing personnel. These factors govern the expenditures on training of personnel (teaching staff, materials) and costs of maintaining personnel (instead of the production of material goods – training).

B. Effect of Lot Size on Cost

First of all, let us dwell on the cost of producing armament specimens. The cost of specimen production is reduced by increasing the quantities of items fabricated. This is a result of a reduction in overhead expenses, the cost of mechanical processing, technical production expenses which, in this case, are distributed over a larger number of items, and through the acquisition by the workers of greater work experience and, consequently, a rise in labor productivity. Moreover, with an increase in production volume material supplies can be improved (large-scale purchases), and it also becomes possible to eliminate structural and production defects.

Cost determination in the USA [11] is frequently accomplished in the assumption that a 20% reduction in average costs per product in a lot can be achieved by doubling the number of items.

Let us derive the appropriate formulas in the assumption that on doubling the number of items the average cost per product in a lot diminishes by a factor of a (according to American data, a = 0.8). Let C_1 denote the cost of the first item; the cost of two items will then be $2aC_1$, the cost of four items will be $4a^2C_1$, the cost of eight items $8a^3C_1$, etc. In the general case we can write that the cost of a lot consisting of n items is

$$C_{\text{партии}} = N a^{\lg_2 N} C_1 = C_1 N^{1+p}, \qquad (1)$$

where

$$P = \frac{\log a}{\log 2}, \qquad (2)$$

for $a = 0{,}8$

$$P \cong -0{,}3.$$

The average cost of an item in a lot from the 1st to the Nth item

$$C_{\text{ср}} = \frac{C_{\text{партии}}}{N} = C_1 N^P. \tag{3}$$

To calculate the cost of the Nth specimen we proceed in the following manner. We find the derivative of C_{partii} from the number of items

$$\frac{\partial C_{\text{партии}}}{\partial N} = (1+P) C_1 N^P. \tag{4}$$

The cost increment

$$\delta C = \left(\frac{\partial C_{\text{партии}}}{\partial N}\right) \delta N. \tag{5}$$

The cost increment for one item ($\delta N = 1$) will be the cost of the item. In this case, instead of N the value of the derivative should show ($N - \frac{1}{2}$), since in this case the value of the derivative will be taken for the mean of the subject interval (from $N - 1$ to N).

Thus

$$C = C_1 (1+P) \left(N - \frac{1}{2}\right)^P. \tag{6}$$

As an illustration of the strong relationship between cost and the number of items, we present the following table.

TABLE 1.11.1

Product number N	1	10	100	1 000
Cost of Nth product, C_N . .	1,00	0,35	0,18	0,09
Average cost per lot, C_{sr} . .	1,00	0,50	0,25	0,13

All previous formulas are valid if in the production process no significant structural changes are introduced into the product, nor any significant changes in the fabrication technology.

C. Determination of Rocket Cost

To determine the tentative cost we can use a weight factor K showing the cost of kilogram of a similar product under the production conditions for a similar quantity of items

$$C_{\text{ср}} = KQ, \tag{7}$$

where Q is the weight of the subject item.

This evaluation exhibits a number of drawbacks associated with differences in design, limited experience in mass production, complexity of calculating costs of measures to maintain rigorous

control and high product quality and, finally, by the introduction of a large number of structural changes calling for technological changes.

The formula cited above is approximate also because it fails to take into consideration the effect of many factors associated with armament cost. Figure 1.11.2 shows the fundamental factors affecting armament cost, which must be taken into consideration and of which we spoke in the previous section.

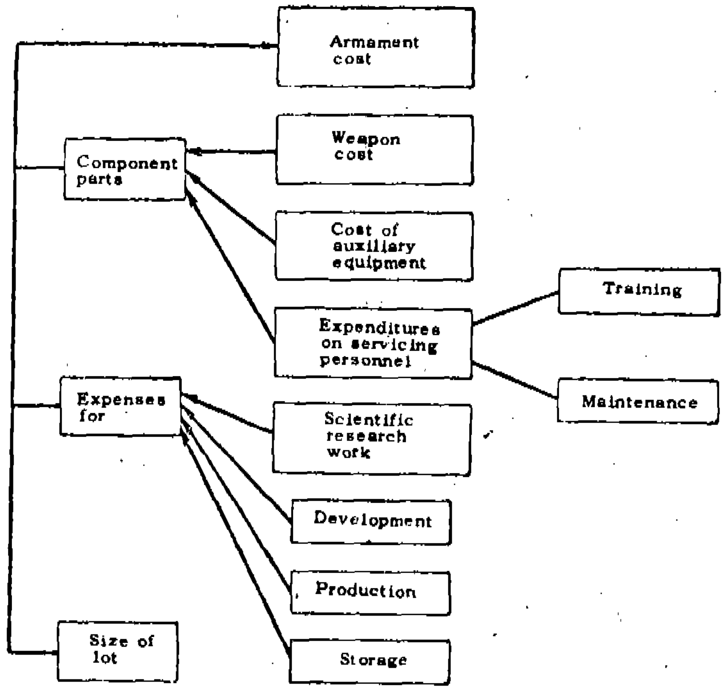

Fig. 1.11.2

At the same time, we can note a feature involving a trend toward a reduction in relative cost as the weight of the rocket increases. This trend exhibits specific physical significance. Obviously, the cost of a large item, all other conditions being equal, will be smaller per unit weight, since the expenditures on production will be reduced in this case.

Analysis shows that this correction is approximately proportional to Q^{-3}. If we take this factor into consideration, the cost of the rocket can be determined from the following formula:

$$C_{cp} = K_1 Q^{2/3}. \quad (8)$$

Finally, even at a given weight the cost of a rocket depends in great measure on design and basic characteristics. Figure 1.11.3 shows a diagram indicating the basic structural features of a rocket and its fundamental characteristics, affecting cost.

However, an approximate method of cost determination for rockets is inapplicable to those cases in which the effect of various factors on cost is to be determined. As an example, let

us consider the derivation of a more detailed formula for the cost of a ballistic rocket.

Let the cost of 1 kg of fuel be denoted C_g. It is a function in great measure of the type of propellant (solid, liquid) and of its energy and operational characteristics.

Let the construction costs for an engine be denoted C_{dv}. It depends in great measure on the type of engine (liquid propellant, solid propellant, ramjet), the number of stages, the materials utilized, the weight characteristics, etc.

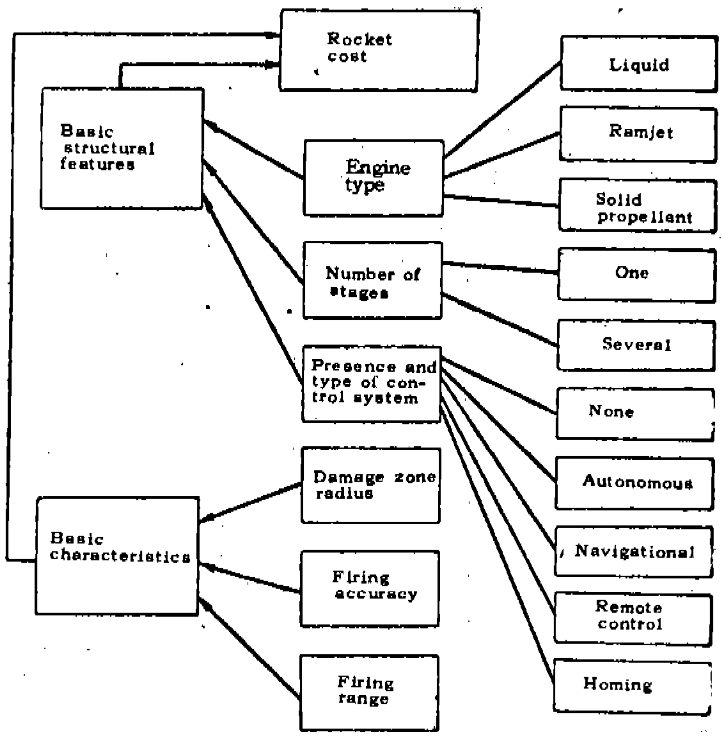

Fig. 1.11.3

Let Q_{su} denote the weight of the control system instrumentation, and let C_{su} denote the cost. The quantity Q_{su} varies in limits from 70-270 kg for autonomous control systems, while cost represents 0.4-0.7 of the cost of the rocket. Let the weight of the warhead be denoted by Q_{bch}, and its cost by C_{bch}. For conventional warheads this cost represents about 10% of the cost of the rocket and is, on the average, equal to 30 dollars per 1 kg.

The cost of the rocket then is

$$C = C_{\text{бч}} + C_{\text{су}} + C_{\text{г}}\omega + C_{\text{дв}}K_{\text{дв}}\omega, \qquad (9)$$

where ω is the weight of the propellant;

K_{dv} is the ratio of the structural weight of the engine to the weight of the propellant.

Proceeding from parabolic theory, we can determine approximately the required velocity v_k to achieve a given range x

$$v=\sqrt{gx}. \tag{10}$$

According to the Tsiolkovskiy formula, this velocity will be equal to

$$v=v_e \ln \frac{\omega+K_{дв}\omega+Q_{су}+Q_{б\,ч}}{K_{дв}\omega+Q_{су}+Q_{б\,ч}}. \tag{11}$$

Here v_e is the effective exhaust velocity.

Hence

$$e^{\frac{v}{v_e}}=\frac{\omega+K_{дв}\omega+Q_{су}+Q_{б\,ч}}{K_{дв}\omega+Q_{су}+Q_{б\,ч}},$$

$$\omega=\frac{(Q_{су}+Q_{б\,ч})\left(e^{\frac{v}{v_e}}-1\right)}{1-K_{дв}\left(e^{\frac{v}{v_e}}-1\right)} \tag{12}$$

or

$$\omega=\frac{(Q_{су}+Q_{б\,ч})\left(e^{\frac{\sqrt{gx}}{v_e}}-1\right)}{1-K_{дв}\left(e^{\frac{\sqrt{gx}}{v_e}}-1\right)}. \tag{13}$$

Substituting (13) into (9) yields

$$C=C_{б\,ч}+C_{су}+$$
$$+\frac{(C_{т}+K_{дв}C_{дв})(Q_{су}+Q_{б\,ч})\left(e^{\frac{\sqrt{gx}}{v_e}}-1\right)}{1-K_{дв}\left(e^{\frac{\sqrt{gx}}{v_e}}-1\right)}. \tag{14}$$

Reference [35] presents the formula associating control system cost with the root mean square error σ of this system

$$C_{су}=\frac{K_{су}}{\sigma^m}. \tag{15}$$

To determine the cost of a high-explosive warhead we can proceed as follows. The weight of the explosive charge (ω_{BB}) is associated in the following manner with the radius $r_з$ of the damage zone:

$$\omega_{BB}=K_r r_з^3. \tag{16}$$

$$Q_{б\,ч}=K_{б\,ч}\omega_{BB}=K_{б\,ч}K_r r_з^3, \tag{17}$$

$$C_{б\,ч}=C_{1б\,ч}Q_{б\,ч}=C_{1б\,ч}K_{б\,ч}K_r r_з^3. \tag{18}$$

Here K_r, K_{bch} and C_{bch} are the coefficients of the damage-zone radius, the quantity of explosive charge in the warhead and cost.

Substituting into (14) the quantities from (15), (17) and (18), we derive a formula associating the cost of a ballistic rocket with its fundamental characteristics: range, accuracy and damage-zone radius:

$$C = C_{1б\,ч} K_{б\,ч} K_r r_3^3 + \frac{K_{оу}}{\sigma^m} + \frac{(C_г + K_{дв} C_{дв})(Q_{оу} + K_{б\,ч} K_r r_3^3)\left(e^{\frac{\sqrt{gx}}{v_e}} - 1\right)}{1 - K_{дв}\left(e^{\frac{\sqrt{gx}}{v_e}} - 1\right)}. \tag{19}$$

This formula has not taken into consideration the cost of the rocket body nor the cost of other units, amounting to 3-10%. Analogous formulas may also be derived for similar rocket types. These, naturally, will be written differently, but the principle of their derivation will not differ from that discussed above.

Cost data for nuclear warheads, taken from [97], are presented below:

TNT equivalent, in thousand tons.......	1.0	10	100	1000
Cost, in millions of dollars...........	2	4	6	8

Approximately these data may be described by the following formula:

$$C_{б\,ч} = 2\sqrt[5]{q}, \tag{20}$$

where q is the TNT equivalent in kilotons;
C_{bch} is the cost of the warhead in millions of dollars.

D. Cost of Ground Equipment

It should be noted that the cost of ground equipment may attain significant magnitudes, exceeding the cost of the rocket severalfold.

We present the cost of American trucks as a function of their capacity.

Table 1.11.3 shows that the cost of a truck may be described approximately by the following formula:

$$C = a + bQ, \tag{21}$$

where a is a coefficient equal approximately to 3000 dollars;
b is the coefficient for the increase in weight with increasing capacity, equal approximately to 1400 dollars per 1 ton;
Q is the capacity.

An analogous formula is valid for helicopters as well. In this case, a = 8000 dollars, b = 190,000 dollars per 1 ton. It goes without saying that in this case it is necessary to take into consideration the relationship between cost and the size of the lot.

Expenses for the construction of silos for ballistic long-range rockets, control centers and appropriate equipment reach significant amounts. This is associated with the need to provide for storage and work areas, equipment for periodic monitoring of the missile complex and for replacement of elements that have broken down or are not functioning sufficiently well. Finally, measures must be taken to ensure safety.

TABLE 1.11.3

Capacity, (t)	Volume of order	Cost of 1 piece, C
0,25	1 000	3 500
0,25	7 500	3 100
0,75	2 400	4 200
0,75	10 000	2 900
2,5	4 200	4 300
2,5	4 200	3 900
5,0	3 200	10 600

Even more substantial expenditures for ground equipment are encountered in the case of antiaircraft missiles. This is associated with the presence of radar stations, complicated computer equipment, these being more expensive than the antiaircraft missiles. In view of this, analysis of ground-equipment cost in a number of cases may prove to be even more important than analysis of rocket cost.

E. Relationship Between the Cost of Development, Production and Storage

In addition to production cost, the cost of development plays an important role. We can see from the data of [44] that this cost is equal, on the average, to the cost of 1000 rockets, i.e., sufficiently large for it to be taken into consideration in evaluating the cost of rocket armament. In addition to development cost, as such, it is necessary also to bear in mind the expenditures associated with production which may involve construction of new factories, the manufacture of new materials, etc.

An important role is also played by expenditures for armament storage. There are indications in the press that in certain cases these expenditures over a period of 10 years may be several times greater than the costs of armament production. Among these expenditures we should include those for the construction of storage areas, their repair, heating, protection and primarily the carrying out of maintenance work and the corresponding replacement of units and components.

Thus, the general expenses for guided missile armament can be determined from the following formula:

$$C_0 = C_{\text{раз}} + C_{\text{пр}} + N_{\text{с}} C_{\text{ср с}} + N_{\text{п}} C_{\text{ср п}} + N_{\text{с}} C_{\text{хр с}} + N_{\text{п}} C_{\text{хр п}} + C_{\text{р}}, \qquad (22)$$

where N_S is the number of missiles;

$C_{\text{sr s}}$ is the average cost of a single missile;

N_{n} is the number of ground systems;

$C_{\text{sr n}}$ is the average cost of the ground system;

$C_{\text{khr s}}$ is the average cost of rocket storage;

$C_{\text{khr n}}$ is the average cost of ground-system storage;

C_{r} represents expenses on training of personnel;

C_{pr} represents the cost of preparation for production.

To calculate the magnitude of $C_{\text{khr s}}$ for 1 year, we can use the following formula:

$$C_{\text{хр с}(1)} = C_{\text{скл с}} + \sum_{i=1}^{m} \frac{C_{\text{ср с}_i}}{a_i} + C_{\text{р}_i} N_{\text{р}_i}, \qquad (23)$$

where $C_{\text{skl s}}$ is the warehousing cost, the equipping and maintenance of warehouses, per 1 rocket per year;

$C_{\text{sr s}_i}$ is the cost of individual units and rocket assemblies (for example, solid propellant charge);

a_i is the depreciation period, i.e., the period within which certain units have outlived their usefulness;

C_{r_1} is the cost of maintaining servicing personnel (security, technical personnel working on replacement of units and assemblies, servicing, monitoring, etc.);

N_{r_1} is the number of people needed to service a single rocket.

According to American data, expenditures for a single military serviceman average 4500 dollars per year.

As a special example, Reference [44] shows the following relationship of expenditures for a total quantity of 7500 missiles:

development cost — 4.3%;

production preparation cost — 8.7%;

missile production cost — 32.5%;

ground equipment production cost — 54%;

storage cost (for 1 year) — 0.5%.

We should bear in mind the great difference between peacetime and wartime economies. If cost is of decisive significance in the former case, time becomes the chief factor in the latter case.

All of the cited figures are extremely approximate and are basically illustrative in nature.

Manuscript Page No.	Transliterated Symbols
33	ф = f = front = front
36	ст = st = statiticheskiy = statistical
36	ц = ts = tsel' = target
37	р = r = raketa = rocket
37	оск = osk = oskolok = fragment(ation)
37	ф = f = fugasnyy = high explosive
37	д = d = detonatsiya = detonation
46	з = z = zaryad = charge (explosive)
46	с = s = snaryad = missile (shell)
50	л = l = letchik = pilot
50	д = d = dvigatel' = engine
50	у = u = upravleniye = control [guidance]
51	к = k = konstruktsiya = structure
51	т = t = toplivnyy otsek = fuel compartment
56	пр = pr = priyemnik = receiver
56	пер = per = peredayushchiy = transmitting
56	ип = ip = ispol'zovaniye ploshchadi = area utilization
61	ср = sr = sredniy = average [mean]
65	оп = op = opyt = experimental [test]
65	н = n = nizko = low
65	в = v = vysoko = high
67	р = r = rabota = work
67	н = n = normal'nyy = normal
67	ф = f = fakticheskiy = actual [factual]
68	п = p = podgotovka = preparation
75	в = v = vklyucheniye = switch on [connection]

75 пр = pr = prirabotka = adjustment

75 г = g = gotovnost' = readiness [preparedness]

76 и = i = ispol'zovaniye = utilization

76 раб = rab = rabota = work

76 рем = rem = remont = repair [overhaul]

76 проф = prof = profilaticheskiy = preventive

81 к = k = kompleks = complex [system]

87 партии = partii = partiya = batch [lot]

90 г = g = goryucheye = fuel [propellant]

90 дв = dv = dvigatel' = engine

90 су = su = sistema upravleniya = control [guidance] system

90 бч = bch = boyevaya chast' = warhead

93 н = n = nazenmyy = ground

93 хр = khr = khraneniye = storage

93 р = r = raschet = team [crew]

93 пр = pr = proizvodstvo = production [manufacture]

94 скл = skl = sklady = warehouses

Chapter 2

THE METHOD OF STATISTICAL TESTS AND ITS APPLICATION IN OPERATIONS RESEARCH

§2.0. INTRODUCTION

The method of statistical tests in operations research, or as it is sometimes known, the Monte Carlo method, is being used extensively. Indeed, the area of applicability for this method has no fundamental limitations and is limited practically only by the time spent on the performance of calculations. Extensive development of electronic computers in considerable measure reduces this limitation and for this reason this is a fundamental method in operations research for the solution of complex problems.

The problems of calculating target damage probabilities are resolved analytically only for comparatively simple cases. In the complex cases which arise in actual practice, the most efficient calculation method is the *method of statistical tests*. The same may be said of problems in the theory of mass service engineering, not to speak of the problems involved in the study of armament effectiveness under difficult combat situations, where consideration must be given to the counterefforts of the enemy, the probabilities of detection, the problems of target distribution, etc.

A positive feature of the method is the simplicity of its practical application. If a subject process can be described by a system of arbitrary equations, rules of logic, etc., the performance of statistical tests presents no basic difficulties, imposes no limitation on the earlier cited equations and rules, nor do these require simplification.

In connection with the material of this chapter, there is a rather detailed description of the essence of the method, its advantages and shortcomings and the most efficient areas of its application are indicated. Methods of deriving random events, function magnitudes and the magnitudes of flows of random events are also described here, as are problems pertaining to evaluation of the accuracy of the method, and certain methods of raising that accuracy.

An example is presented at the end of this chapter on application of the method of statistical tests to determine guided-missile firing accuracy.

§2.1. THE METHOD OF STATISTICAL TESTS AND THE AREAS OF ITS APPLICATION

A. The Essence of the Method of Statistical Tests

The method of statistical tests (or the Monte Carlo method) presently refers to any procedure involving the use of methods of statistical selection for approximate solutions of a given mathematical or physical problem. We will immediately point to two variations in the utilization of this method:

the application of the method of statistical tests for a study of probability processes (it is assumed in this case that the mathematical or physical model of this process has been developed);

the application of the method of statistical tests for solution of nonprobability problems (by analogy between the equations which describe this problem and those which describe the probability processes).

In operations research it frequently becomes necessary to resort to the first variant of this method. In studying military action it is comparatively easy to make up complex descriptive schemes (models) including the probabilities associated with such random elements as detection probability of certain targets, the probability of reliability in operation for all system elements, the probability of hitting a target, the probability of damaging a rocket in flight, the probability of target destruction, etc. The study of such a model by analytical methods is extremely difficult in many cases. In any event, the possibilities of constructing complex probability models describing the processes of military action rapidly overtake the potentials of mathematical analysis for the study of such models. In this connection, the methods of statistical tests in a number of cases have turned out to be the only practically suitable methods of study. We should take note of the fact that the majority of models in which we examine bilateral military action (counteraction) are exceedingly complex for investigations by analytical methods and in these cases it becomes necessary to resort to the method of statistical tests.

Let us examine the application of the method of statistical tests on the simplest of examples. Given the calculation of the probability (P) of a rocket hitting a target represented by a circle of radius r, for the case of rocket scattering characterized by the quantities $\sigma_x = \sigma_z = \sigma$, and no systematic error.

This problem is solved analytically (see §3.3)

$$P = \int_0^{R_1} Re^{-\frac{R^2}{2}} dR = 1 - e^{-\frac{R_1^2}{2}}, \qquad (1)$$

where

$$R_1 = \frac{r}{\sigma}. \qquad (2)$$

As an example this problem has been chosen for convenience in comparison of exact results with the method of statistical tests. At the same time, slightly greater complexity in this problem makes it impossible of exact analytical solution, and the method of statistical tests for practical cases thus becomes expedient.

To calculate P by the method of statistical tests it is necessary to go through a series of tests:

1. Determine the coordinates x_n and z_n of the points of rocket impact

$$x_n = \sigma\delta_{1,n}, \tag{3}$$

$$z_n = \sigma\delta_{2,n}, \tag{4}$$

where $\delta_{1,n}$ and $\delta_{2,n}$ are random numbers distributed normally with root mean square errors equal to unity and a mathematical expectation equal to zero. We will speak of the methods for the derivation of these numbers in §2.3. In this case we will use a table of random numbers (see Table 2 of the appendix);

n is the realization number (of the test).

TABLE 2.1.1

n	1	2	3	4	5	6	7
$x_n = \delta_1$	0.80	−0.54	0.42	−0.48	0.16	1.95	1.87
$z_n = \delta_2$	−0.67	0.61	1.15	−0.19	−0.90	−0.70	−0.36
r_n	1.04	0.81	1.22	0.52	0.92	2.07	1.90
m	0	1	1	2	3	3	3
P	0	0.50	0.33	0.50	0.60	0.50	0.43

n	8	9	10	11	12	13	14
$x_n = \delta_1$	0.63	−1.48	−0.49	−2.92	1.72	−0.90	−0.24
$z_n = \delta_2$	0.05	0.56	1.28	−1.18	−0.66	−0.68	1.76
r_n	0.63	1.58	1.37	3.15	1.84	1.13	1.78
m	4	4	4	4	4	4	4
P	0.50	0.44	0.40	0.36	0.33	0.31	0.29

n	15	16	17	18	19	20
$x_n = \delta_1$	0.24	0.34	−0.88	−1.07	0.47	1.46
$z_n = \delta_2$	−2.47	−0.32	2.22	0.02	−0.55	2.62
r_n	2.48	0.47	2.39	1.07	0.72	3.00
m	4	5	5	5	6	6
P	0.27	0.31	0.29	0.28	0.32	0.30

2. Calculate the distance between the points of rocket impact and the target

$$r_n = \sqrt{x_n^2 + z_n^2}. \qquad (5)$$

3. Compare r_n and r. If $r_n < r$, we have a hit on the target. Let there be m such cases. If $r_n > r$, there is no hit.

4. Calculate the probability of a target hit

$$P = \frac{m}{n}. \qquad (6)$$

The results obtained from calculations by this method are presented in Table 2.1.1 for the case $\sigma = 1$ and $R_1 = 1$.

The exact value of probability $P = 0.393$. The table shows that 20 tests permit determination of this quantity with insufficiently high accuracy. The error is about 25% and the root mean square deviation, calculated with Formula (2.3.12), amounts to 27.5% of the determined quantity. It is interesting to note that with three realizations ($n = 3$) a more exact result is obtained than with 20 realizations, while an even more exact result is obtained with 10 realizations. These, however, are random facts. The root mean square deviation of the derived results from the true value with an increase in the number of realizations, as will be demonstrated below, diminishes in a regular manner.

B. Application of the Method of Statistical Tests for the Solution of Nonprobability Problems

Since the following chapters will present and consider in comparative detail a number of examples for the application of the method of statistical tests in the study of probability models associated primarily with analysis of military action, we note other possible applications of this variation of the method. This method may be employed to solve an entire series of problems associated with the study of the operation of an individual armament model, in particular for the evaluation of the reliability exhibited by complex systems, to study the firing accuracy of rocket systems, to study production operations associated with fabrication as well as with the assembly of rockets and the preparation of these for launching, etc.

The application of the method of statistical tests for the solution of nonprobability problems is associated, first of all, with the construction of probability model-analogs of functional equations. This method is used most extensively to calculate integrals, and particularly, multiple integrals.

To calculate the integral $\int_a^b f(x)\,dx$ we can use the following method. To calculate this integral means to determine the area that is cross-hatched in Fig. 2.1.1. We select random pairs of numbers $\alpha_1\beta_1$, $\alpha_2\beta_2$, etc., in which α_i are distributed according to the

law of equal probability in the interval a, b, and in which β_i is distributed according to the law of equal probability in the interval 0, y, and we will observe to see into which area the point falls: the cross-hatched area, as $\alpha_1\beta_1$, $\alpha_3\beta_3$, or in the area that is not cross-hatched, as $\alpha_2\beta_2$, $\alpha_4\beta_4$. This indicates the satisfaction or nonsatisfaction of the inequality

$$f(\alpha_i) > \beta_i.$$

Let the number of cases in which this inequality is satisfied, i.e., when the point falls in the cross-hatched area, be equal to m, and let all of the cases be denoted by n. It is then proved that

$$\int_a^b f(x)\,dx = \frac{m}{n} y(b-a). \tag{7}$$

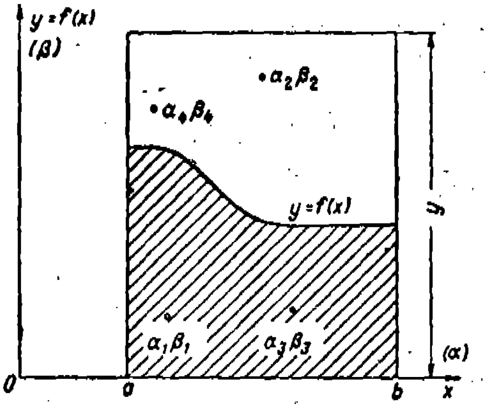

Fig. 2.1.1

It is obvious that if the points are uniformly distributed over the area, the number of their inclusions within the area bounded by the curve $y = f(x)$, the axis of abscissas and the verticals $x = a$ and $x = b$ will pertain to the total number of tests as the area of the above-indicated region pertains to the entire area into which the points may fall. From this we have Formula (7).

Another method is also possible. Let α denote a quantity uniformly distributed over the interval (a, b). If we select a specific realization number (n) of this quantity α_i, if we calculate $f(\alpha_i)$ and the mathematical expectation

$$M[f(\alpha_i)] = \frac{1}{n}\sum_{i=1}^{n} f(\alpha_i), \tag{8}$$

it is proved that

$$\int_a^b f(x)\,dx = (b-a)\,M[f(\alpha_i)]. \tag{9}$$

Indeed, the mathematical expectation $f(x)$ in the interval (a, b)

$$M[f(x)] = \frac{\sum_{i=1}^{n} f(x)}{n}$$

or at the limit $(n \to \infty)$

$$M[f(x)] = \frac{\int_{a}^{b} f(x)\,dx}{b-a},$$

from which Formula (8) follows.

A large number of simple similar calculations is characteristic of both cases.

EXAMPLE. Calculate $J = \int_{0}^{R_1} Re^{-\frac{R^2}{2}} dR$ for $R_1 = 1$. We note the exact value of this integral $J = 1 - e^{-\frac{R^2}{2}}$. With $R_1 = 1$, $J = 0{,}393$. As an example this quantity has been selected for convenience in evaluating result accuracy. Let us now use the method of statistical tests, and primarily, the first method. For this, from the table of random numbers distributed according to the law of equal probability (Table 2 of the appendix, we select 2 numbers each, beginning with 41 in sequence (δ_1 and δ_2) and, having multiplied these by 10^{-5}, we round off to 0.001. We then calculate $f(\delta_1)$ and compare with δ_2.[1] If $f(\delta_1) > \delta_2$, we hold that the event occurred; in this opposite case, the event did not occur. Summing the number of cases in which the event occurred, we refer these cases to the number of experiments and derive the quantity $\mathcal{J}_1$.[2] Table 2.1.2 shows the calculational results for a test number up to 20. For each number of trials the value of the integral has been calculated and denoted $\mathcal{J}_1$.

When using the second method we seek only the random number δ_1, we calculate $f(\delta_1)$ and then the integral according to Formula (9). This value is known as $\mathcal{J}_2$ and also shown in the table. That table also shows the square roots from the selected dispersions of $\mathcal{J}_2$, which are denoted S, and the values of σ are also calculated from Formula (2.3.2).

The first calculation method yielded an error of 14.5% and the second calculation method yielded a smaller error of only 7.9%. The root mean square deviation in the first case, calculated with Formula (2.3.12), amounts to 27.5%; in the second case, calculated with Formula (2.3.8), it amounts to 9.4%. Thus the second method for the calculation of the integral is more effective.

Methods analogous to those indicated above may be suitable for solution of systems of linear equations, solutions of boundary problems, the calculation of continuous integrals and a whole series of other problems. However, since these calculations in operations research are only auxiliary in nature, we will not dwell on these any longer.

[1]/163
[2]/163

TABLE 2.1.2

n	1	2	3	4	5	6	7
δ_1	0,667	0,993	0,242	0,940	0,610	0,131	0,352
δ_2	0,491	0,182	0,192	0,025	0,557	0,530	0,865
$f(\delta_1)$	0,534	0,607	0,235	0,604	0,506	0,130	0,331
m	1	2	3	4	4	4	4
$\mathcal{J}_1$	1	1	1	1	0,800	0,667	0,571
$\mathcal{J}_2$	0,534	0,570	0,459	0,495	0,497	0,436	0,421
S	—	0,0548	0,195	0,176	0,153	0,203	0,190
σ	—	—	—	0,1520	0,0967	0,1070	0,0880

n	8	9	10	11	12	13	14
δ_1	0,646	0,646	0,680	0,398	0,339	0,806	0,699
δ_2	0,105	0,564	0,136	0,579	0,541	0,238	0,432
$f(\delta_1)$	0,525	0,525	0,540	0,368	0,320	0,583	0,547
m	5	5	5	6	6	7	8
$\mathcal{J}_1$	0,625	0,556	0,600	0,545	0,500	0,538	0,571
$\mathcal{J}_2$	0,434	0,444	0,454	0,446	0,435	0,447	0,454
S	0,181	0,171	0,163	0,157	0,157	0,152	0,150
σ	0,0757	0,0658	0,0584	0,0529	0,0501	0,0463	0,0436

n	15	16	17	18	19	20
δ_1	0,984	0,327	0,129	0,146	0,669	0,430
δ_2	0,033	0,462	0,284	0,161	0,990	0,547
$f(\delta_1)$	0,606	0,310	0,128	0,145	0,535	0,392
m	9	9	9	9	9	9
$\mathcal{J}_1$	0,600	0,562	0,529	0,500	0,474	0,450
$\mathcal{J}_2$	0,464	0,454	0,435	0,419	0,425	0,424
S	0,151	0,151	0,166	0,173	0,171	0,165
σ	0,0421	0,0405	0,0431	0,0435	0,0416	0,0390

C. Application of the Methods of Statistical Tests for the Solution of Probability Problems

With respect to the solution of the first group of problems, i.e., the study of probability processes, the method may be divided into the following basic stages (Fig. 2.1.2):

1) determination of the characteristics of random processes (initial data);

2) the derivation of realizations of random numbers, functions, flows and events;

3) the performance of multiple repetitive calculations with respect to a selected algorithm describing the probability model of the subject process, proceeding from random realizations selected in the previous item;

4) statistical processing of results, evaluation of accuracy for results and making decisions as to cessation or continuation of the process of statistical tests. An important position is held by the human monitoring of model operation. In certain

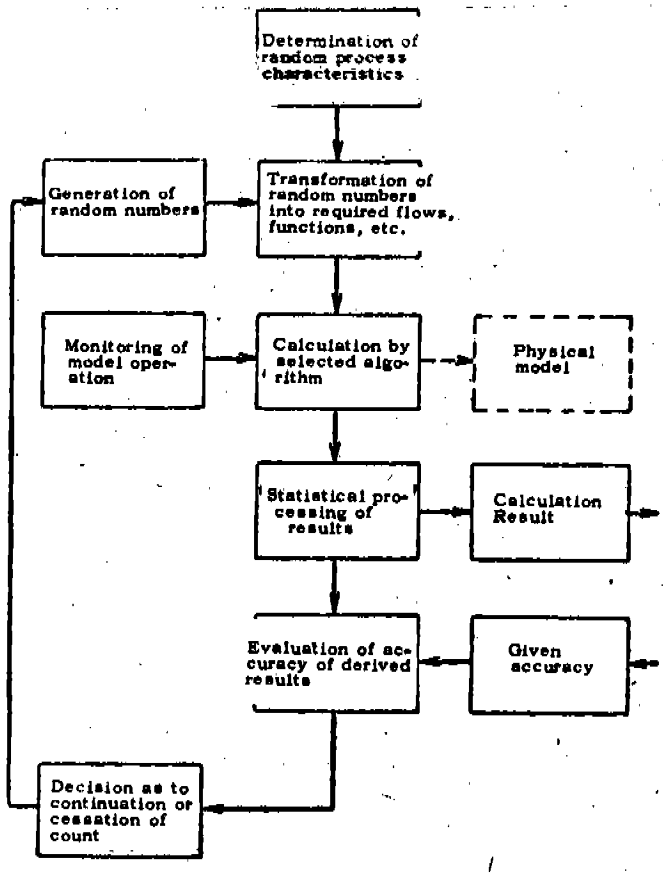

Fig. 2.1.2

cases it becomes necessary to combine the mathematical model with a physical model. There then arises the question of the combination of the mathematical model with instruments, devices or even with a human being.

To study any probability process we have to know the characteristics of the random quantities, functions, flows, etc., defining the process. For models of military action these characteristics include the probabilities of target detection, faultless weapon operation, hitting the target, damaging the target, etc., i.e., the characteristics of weapons systems. As a rule, these characteristics are determined experimentally. In a number of cases they may be assumed in order to determine their optimum levels. Finally, they can be determined theoretically, in particular, by the same method of statistical tests in which we use the characteristics of individual weapons elements that have been determined experimentally.

In addition to the characteristics of weapons sites, in a number of cases we must consider the random characteristics of the ambient medium, as, for example:

of the atmosphere (temperature, pressure, direction and speed of wind, visibility from various altitudes, and at various distances, etc.);

relief (direct visibility range, the availability of cover, the possibility of movement, etc.).

The location of combat unit elements (the distance between units at the front and in the rear) and the strength of the units (with consideration of losses), the preparation [training] time, the speed of movement, etc., may all be random quantities.

Determination of all of these characteristics goes beyond the limits of the method of statistical tests; however, it is important to bear in mind that the accuracy requirements imposed on the method must be set on the basis of the accuracy with which these characteristics are determined. The more coarsely these characteristics are determined, the less rigid the requirements that are to be imposed on the accuracy of the method.

D. The Accuracy of the Method and the Complexity of the Model

Several words should be said in connection with the accuracy of the quantity derived in the process of statistical tests and its relationship to the root mean square deviation of the random quantities affecting the process and their mutual relationship.

In the course of the solution we determine the mathematical expectation M which is a sum of k independent quantities x_i with the mathematical expectation M_i and the root mean square deviation σ_i. For example, let this be the mathematical expectation of the number of targets damaged by various means. The quantity σ for M in a single test will then be

$$\sigma=\sqrt{\sum_{i=1}^{k}\sigma_i^2}, \tag{10}$$

$$M=\sum_{i=1}^{k}M_i. \tag{11}$$

The relative magnitude of σ

$$\frac{\sigma}{M}=\frac{\sqrt{\sum_{i=1}^{k}\sigma_i^2}}{\sum_{i=1}^{k}M_i}. \tag{12}$$

Let

$$M_1=M_2=\ldots=M_i=M_1,$$

$$\sigma_1=\sigma_2=\sigma_3=\ldots=\sigma_i=\sigma_1.$$

Then

$$\frac{\sigma}{M}=\frac{\sigma_1}{M_1\sqrt{k}}. \tag{13}$$

Thus, with an increase in the number of independent factors affecting the process, the relative magnitude of the mean square deviation of M diminishes. If the quantities x_i are dependent,

according to page 213 in Reference [25]

$$\sigma = \sqrt{k\sigma_1^2[1+r(k-1)]}, \tag{14}$$

where r is the correlation factor (we assume that they are all equal).

In this case the relative magnitude of the root mean square deviation diminishes more slowly, but nevertheless diminishes. This means that the method of statistical tests yields a more exact expectation of the sought quantity, the more complex the process being studied (in the sense of the number of affecting factors). Of course, the foregoing cannot serve as a rigorous proof of this statement; however, experience in the study of complex probability processes convinces us of the fact that with an increase in the number of random factors affecting the process, the application of the method of statistical tests becomes more feasible.

Let us examine the situation in which the method of statistical tests is used to determine the root mean square deviation of some quantity.

Given that we are seeking the root mean square deviation of the quantity z which is a sum k of the quantities y: $z=\sum_{i=1}^{k} y_i$. We assume that $\sigma_{y_1}=\sigma_{y_2}=\sigma_{y_3}=\ldots=\sigma_y$.

Then $\bar{z}=\sum_{i=1}^{k}\bar{y}_i$ and according to (14)

$$\sigma_z^2 = k\sigma_y^2 + k(k-1)r\sigma_y^2.$$

After N tests, the square of the selected root mean square deviation

$$S_z^2(N-1)=\sum_{j=1}^{N}(z_j-\bar{z})^2=\sum_{j=1}^{N}\left(\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}\bar{y}_i\right)^2=$$

$$=\sum_{j=1}^{N}\left[\sum_{i=1}^{k}(y_i-\bar{y}_i)\right]^2.$$

However, according to the rule for the calculation of the square of the sum

$$\left[\sum_{i=1}^{k}(y_i-\bar{y}_i)\right]^2=\sum_{i=1}^{k}(y_i-\bar{y}_i)^2+2\sum_{\substack{i=1\\l=1}}^{k}(y_i-\bar{y}_i)(y_l-\bar{y}_l).$$

Then

$$S_z^2(N-1)=\sum_{i=1}^{k}\sum_{j=1}^{N}(y_i-\bar{y}_i)^2+2\sum_{\substack{i=1\\l=1}}^{k}\sum_{j=1}^{N}(y_i-\bar{y}_i)(y_l-\bar{y}_l).$$

Taking into consideration that

$$\sum_{j=1}^{N}(y_i-\bar{y}_i)^2=(N-1)S_{y_i}^2,$$

$$\sum_{j=1}^{N}(y_i-\bar{y}_i)(y_l-\bar{y}_l)=(N-1)m_{y_iy_l},$$

where $m_{y_iy_l}$ is the correlation moment. Then we obtain

$$S_z^2=\sum_{i=1}^{k}S_{y_i}^2+2\sum_{\substack{i=1\\l=1}}^{k}m_{y_iy_l}.$$

We know [36] that after N tests

$$\sigma^2(S_y^2)=\frac{2\sigma_y^4}{N-1};\ \sigma^2(m_{y_iy_l})=\frac{1-r^2}{N-1}\sigma_{y_i}^2\sigma_{y_l}^2.$$

Assuming all σ_y to be identical, we obtain

$$\sigma^2(S_z^2)=k\sigma^2(S_y^2)+k(k-1)\sigma^2(m_{y_iy_l})=$$
$$=\frac{2k\sigma_y^4}{N-1}+\frac{2k(k-1)}{N-1}(1-r^2)\sigma_y^4.$$

Hence

$$\sigma(S_z^2)=\frac{\sigma_y^2\sqrt{2k}}{\sqrt{N-1}}\sqrt{1+(k-1)(1-r^2)},$$

$$\frac{\sigma(S_z^2)}{\sigma_z^2}=\sqrt{2}\ \frac{\sqrt{1+(k-1)(1-r^2)}}{\sqrt{k(N-1)}[1+(k-1)r]}.$$

Let the quantities y not be related. Then $r = 0$,

$$\frac{\sigma(S_z^2)}{\sigma_z^2}=\frac{\sqrt{2}}{\sqrt{N-1}}, \tag{15}$$

i.e., the root mean square deviation of the selected dispersion is independent of the number of factors affecting the process (k).

Let the quantities y be associated so that $r = 1$. Then

$$\frac{\sigma(S_z^2)}{\sigma_z^2}=\frac{\sqrt{2}}{k\sqrt{k(N-1)}}, \tag{16}$$

i.e., the root mean square deviation in this case for the sought dispersion drops sharply with the increasing number of factors affecting the dispersion.

Thus the highest accuracy in the determination of the root mean square deviation can be expected from the method of statistical tests in those cases in which it is defined by mutually

associated quantities, i.e., precisely in those cases in which the analytical solution is most difficult.

The considerations presented above are primarily illustrative in nature; however, they may be useful in selecting methods for the study of models.

E. Area of Applicability for the Method

The law of large numbers serves as the theoretical basis for the method of statistical tests. The Chebyshev theorem, a form of this law, states that with an unlimited increase in the number of independent tests the arithmetic mean of the observed values of a random quantity exhibiting a finite dispersion converges in probability to its mathematical expectation, i.e.,

$$\lim P\left(\left|\frac{\sum_{i=1}^{n} x_i}{n} - m_x\right| < \varepsilon\right) = 1 \qquad (17)$$

for any $\varepsilon > 0$, where x_i are independent random quantities with the mathematical expectation m_x and finite dispersion.

The Bernoulli theorem, another form of this law, states that with an unlimited increase in the number of independent tests under constant conditions the frequency of the subject event converges in probability to the probability of that event, i.e.,

$$\lim P(|f_n(A) - P| < \varepsilon) = 1, \qquad (18)$$

where A is the event, $f_n(A)$ is the frequency of that event in n tests and P is the probability of that event.

Thus the method of statistical tests is based on the most general theorems of the theory of probabilities and essentially contains no limitations. This method may be applied to the solution of any problem, and with a sufficiently large number of tests any degree of accuracy may be imposed on the method.

These circumstances represent incomparable advantages of the method and govern its extensive application for solution of the most varied and most complex problems. This method is frequently used to evaluate approximation theories, acting in the role of unique experiment.

At the same time, this method exhibits the drawback of high labor input, in which connection extensive application of this method began with the development of computer engineering. It is true that in specific cases this difficulty is not great, as it may appear at first glance, and in a number of cases it can be reduced substantially.

Metaphorically, the second drawback of the method is its "blindness." As the method is applied there is no way of seeing how certain factors affect the derived results and it therefore

becomes necessary to carry out a large number of calculations even for a qualitative study of the effect of various factors, which is important in design procedures, selection of optimum solutions, etc., whereas analytical methods make it possible to carry out such evaluations simply.

The combined application of simplified analytical methods making possible the selection of a comparatively narrow range of studies, the evaluation of the nature of the effect of various factors, the simplification of the model of a phenomenon by elimination of secondary factors, and the method of statistical tests making possible evaluation of a more exact but more limited region is therefore the most efficient solution.

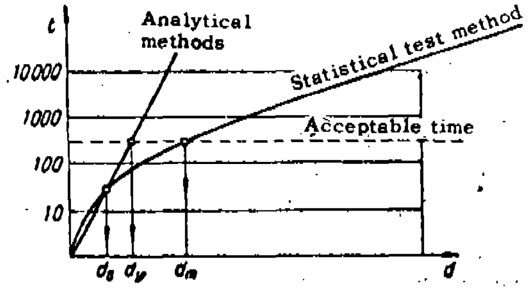

Fig. 2.1.3

It is precisely this combination of analytical and statistical-test methods that is recommended for operations research. In the literature [114] we find the following interesting curve showing the expenditure of time (t) on the solution of a problem by the analytical method and by the method of statistical tests as a function of the number of independent parameters affecting the subject process (d).

This curve is shown in Fig. 2.1.3. The denotations here are the following:

d_e is the number of factors from which the method of statistical tests becomes most advantageous;

d_φ is the number of factors from which the application of analytical methods requires more time than is acceptable;

d_m is the number of factors from which the method of statistical tests requires more time than is acceptable.

We can see from this graph that the study of complex processes is feasibly carried out by the method of statistical tests, while simple processes are most advantageously studied by analytical methods.

We should take note of the fact that the method of statistical tests has much in common with experiments. As in the case

of experiment, we obtain a final result that is sufficiently reliable and, as a rule, requires considerable expenditure of work. Its drawback in comparison with experiment is the impossibility of taking into consideration all factors. However, it exhibits an advantage consisting in the possibility of studying a process which is difficult to study by experimentation. For example, the derivation of limit values for weather conditions in tests carried out under actual conditions is difficult but simple with simulation; firing at arbitrary azimuths over sufficiently large ranges is difficult, but rather simple in simulation.

§2.2. DETERMINATION OF RANDOM NUMBERS AND FORMATION OF REALIZATIONS IN THE SIMPLEST CASES

For the calculation of each realization of a sought quantity by application of the method of statistical tests we have to achieve random realizations of the quantities, functions, flows and simplest phenomena.

The starting point for the formation of any of these realizations in discrete calculation is a device exhibiting equal probability of yielding 0 or 1. A coin may serve this purpose, with the index 0 on one side of the coin, and unity on the other; chips in an urn (marked 0 and 1 equally); a die with three sides marked 0 and the other three sides marked 1; and special electronic devices.

Having achieved appropriate amounts of 0 and 1 in random sequence, from these we can form random numbers distributed according to the law of equal probability and exhibiting an arbitrary number of signs. The methods of this formation are described in Item A.

Moreover, we can use tables of appropriate random numbers calculated by someone in advance. Using uniformly distributed numbers, we can obtain numbers distributed according to other laws (Item B), realizations of random functions (Item C), of flows (Item D) and of events (Item E).

If the experiment yielded realizations of random quantities, functions, flows, etc., having assigned a number to each such realization and using uniformly distributed random numbers in the interval from 0 to the number of realizations at our disposal, we can select the appropriate realizations of random quantities, functions, flows, etc.

A. Derivation of Uniformly Distributed Random Numbers

As a rule, random numbers are distributed according to the law of equal probability in the range 0-1. There are special tables of such quantities, an example of which is given in the appendix (see Table 2, showing random quantities distributed uniformly in the interval 0-99999). Such tables are used in manual calculations. Basically speaking, these may be used in computer calculations. However, the large quantity of such numbers needed for calculations, in addition to the limited memories of the computers, compels us to employ other methods of deriving the quan-

tities, and of these it is important to take note of two: by means of physical generators of random numbers and by means of methods for the derivation of pseudorandom numbers.

The simplest physical generator of random numbers is a coin with 0 indicated on one side and 1 shown on the other. With this generator we can have the numbers 0 and 1 with equal probability. The random number is calculated according to the formula

$$\xi = z_1 \cdot 2^{-1} + z_2 \cdot 2^{-2} + z_3 \cdot 2^{-3} + \ldots + z_n \cdot 2^{-n} + \ldots, \qquad (1)$$

where z_n is the sequence of numbers 0, 1 derived by means of the generator described above.

It is not difficult to prove that the quantity ξ is distributed uniformly from 0 to 1. However, we have to limit the number of bits in the practical realization of this method. In an electronic computer this is associated with design (the number of bits), in manual procedures this is associated with the need to save time. Therefore, instead of a continuous population of numbers with uniform distribution, we use a discrete population of numbers. If the computer has k binary bits, this population will consist of 2^k numbers with identical appearance probabilities. The number of binary bits in existing computers is so great (around 30) that there is no reason to expect any inaccuracies because of discreteness in the solution of problems pertaining to operations research.

As an example let us consider the calculation of the random number ξ from the uniform distribution of 0,1, limiting ourselves to 7 bits. Tossing a coin, we obtained the following sequence: z = 0, 1, 1, 0, 1, 0, 1. On the basis of (1) we then obtain

$$\xi = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128} = 0{,}41.$$

The calculation result was rounded off to 0.01, since this is approximately the magnitude of the discreteness spacing in the case of 7 binary bits (1/128).

Practically, radioactive particle counters or radio tubes with significant tube noise are used as the physical generators.

The first method involves the following. If an even number of particles has been counted at a precise instant of time, this corresponds to a reading of 1; otherwise, it corresponds to 0. The circuit of such a generator is easily connected to a computer.

In the second case the radio tube noises are converted into a series of pulses which play the role of the radioactive particles in the previous generator.

Pseudorandom numbers are numbers derived by calculation with a special algorithm without resort to physical generators. One of the possible methods of achieving uniformly distributed pseudorandom numbers (the means of a square) involves the following.

Given an ξ_{n-1} — m-bit binary number

$$\xi_{n-1} = \varepsilon_1 \cdot 2^{-1} + \varepsilon_2 \cdot 2^{-2} + \ldots + \varepsilon_m \cdot 2^{-m}. \tag{2}$$

The square of this number will have the form

$$\xi^2_{n-1} = \delta_1 \cdot 2^{-1} + \delta_2 \cdot 2^{-2} + \ldots + \delta_{2m} \cdot 2^{-2m}. \tag{3}$$

Let us isolate the mean bits of this number and assume

$$\xi_n = f(\xi_{n-1}) = \delta_{\frac{m}{2}+1} \cdot 2^{-1} + \delta_{\frac{m}{2}+2} \cdot 2^{-2} + \ldots + \delta_{\frac{3m}{2}} \cdot 2^{-m}. \tag{4}$$

Analogously we can derive ξ_{n+1}, etc.

Let us take as an example

$$\xi_{n-1} = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} \cong 0{,}41.$$

Squaring this by successive multiplication, we obtain

$$\begin{aligned}
\xi^2_{n-1} = \frac{0}{4} + \frac{0}{8} + \frac{0}{16} + \frac{0}{32} + \frac{0}{64} &+ \\
+ \frac{0}{8} + \frac{1}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128} &+ \\
+ \frac{0}{16} + \frac{1}{32} + \frac{1}{64} + \frac{0}{128} + \frac{0}{256} &+ \\
+ \frac{0}{32} + \frac{0}{64} + \frac{0}{128} + \frac{0}{256} + \frac{0}{512} &+ \\
+ \frac{0}{64} + \frac{1}{128} + \frac{1}{256} + \frac{0}{512} + \frac{1}{1024} & \\
\hline
\xi^2_{n-1} = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128} + \frac{0}{256} + \frac{0}{512} + \frac{1}{1024} &.
\end{aligned}$$

As ξ_n we assume the following number (because in our example m is odd, instead of the resulting $(m/2) + 1 = 3.5$, we take 3, etc.):

$$\xi_n = \frac{1}{2} + \frac{0}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} \cong 0{,}66.$$

The danger encountered with such methods is "degeneration" (getting zeros in all bits, the formation of cycles of repeated sequences, etc.).

Let us note that on obtaining pseudorandom numbers it is absolutely necessary to carry out a statistical verification of these numbers (of the mathematical expectation, dispersion, and the distribution function).

B. Derivation of Random Quantities Distributed According to Arbitrary Laws

Let a random number λ_i be derived from a uniform distribution from 0 to 1. This has to be transformed into a random number from a sequence with a given distribution function. There are

2 basic methods for this transformation: using the properties of an integral distribution function and the properties of the distribution laws.

We examine the first method. In the theory of probabilities [71] it is proved that if the random quantity ξ has the distribution density $f(x)$, the distribution of the random quantity

$$\eta = \int_{-\infty}^{\xi} f(x)\,dx \tag{5}$$

is uniform from 0 to 1.

Proceeding from this property, the rule for the derivation of numbers x_i distributed according to the given law $f(x)$ will be the following. We obtain λ_i from a uniform distribution in the interval 0-1. We find the quantity x_i by solving the equation

$$\lambda_i = \int_{-\infty}^{x_i} f(x)\,dx. \tag{6}$$

As first example we examine the transformation of the quantity λ_i into x_i distributed with probability density $f(x) = 2x$ in the interval 0-1. On the basis of (6) we write

$$\lambda_i = \int_0^{x_i} 2x\,dx, \tag{7}$$

$$\lambda_i = x_i^2. \tag{8}$$

Hence

$$x_i = \sqrt{\lambda_i}.$$

As second example we present the formula for the derivation of the random numbers ξ_i distributed according to the exponential law

$$f(\xi) = \lambda e^{-\lambda\xi}. \tag{9}$$

On the basis of (6) we write

$$\lambda_i = \int_{-\infty}^{\xi_i} \lambda e^{-\lambda\xi} d\xi = 1 - e^{-\lambda\xi_i}. \tag{10}$$

Hence

$$\xi_i = -\frac{1}{\lambda}\ln(1-\lambda_i).$$

Having taken into consideration that $1 - \lambda_i$ is distributed accordint to the law of equal probability, as is λ_i, we can simplify this formula

$$\xi_i = -\frac{1}{\lambda}\ln\lambda_i.$$

We can proceed analogously to find the numbers x_i distributed according to the Weibull law. For this law

$$\int_0^{x_i} f(x)\,dx = 1 - e^{-\frac{(x_i - x_n)^m}{x_0}}. \tag{11}$$

Then

$$\lambda_i = 1 - e^{-\frac{(x_i - x_n)^m}{x_0}}. \tag{12}$$

Here x_n, m and x_0 are the parameters of the Weibull law; λ_i are the uniformly distributed numbers in the interval 0-1.

The formula for the derivation of the numbers r_i distributed according to the Rayleigh law is derived analogously:

$$r_i = \sigma\sqrt{2\ln(1-\lambda_i)}. \tag{13}$$

Here σ is the parameter of the Rayleigh law.

This is a universal method. However, in a number of cases it cannot be used conveniently on a computer. This occurs when it is impossible to take the integral and the table for the integral distribution function has to be introduced into the computer memory. Then it is possible to use the second method which involves utilization of specific features of the laws.

Given that we have to find a number distributed normally. We must either utilize tables or the integral distribution function, or to recall one of the basic properties of the normal law, involving the fact that the sum of a sufficiently large number of numbers distributed according to arbitrary laws will be distributed according to the normal law.

We will calculate the numbers x_i by means of the formula

$$x_i = \sum_{j=1}^{n} \lambda_j, \tag{14}$$

where λ_j are distributed uniformly from 0 to 1. According to the foregoing, the numbers x_i will then be distributed according to the normal law. The mathematical expectation and the root mean square deviation of the quantities x_i will be

$$M(x_i) = \frac{n}{2}. \tag{15}$$

$$\sigma(x_i) = \frac{1}{2\sqrt{3}}\sqrt{n}. \tag{16}$$

If the required number y_i must correspond to a normal sequence with $M(y_i) = a$ and $\sigma(y_i) = \sigma$, the resulting number x_i must be transformed in the following way:

$$y_i = \frac{\left(x_i - \frac{n}{2}\right)\sigma}{\frac{1}{2\sqrt{3}}\sqrt{n}} + a. \tag{17}$$

Practically, the normal distributed random quantities are derived in this way. However, for close correspondence to normal distribution the number of terms must be sufficiently large (on the order of 10). To reduce this number we employ the following transformation:

$$\alpha_i = \frac{x_i}{\sqrt{n}} - \frac{1}{20n}\left(\frac{2x_i}{\sqrt{n}} - \frac{x_i^3}{n\sqrt{n}}\right). \tag{18}$$

The distribution law for α_i with $n = 5$ is sufficiently close to the normal. A transformation of the following type [12]

$$\alpha_i = \frac{x_i}{\sqrt{n}}\left[1 - \frac{41}{13440n^2}\left(\frac{x_i^4}{n^2} - 10\frac{x_i^2}{n} + 15\right)\right] \tag{19}$$

makes it possible to achieve good agreement with the normal distribution when $n = 2$.

For the second example of the utilization of this method, let us consider the derivation of the quantities distributed according to the Rayleigh law. We know that the modulus of the random vector is distributed according to the Rayleigh law. Hence follows the rule for the derivation of numbers distributed according to the Rayleigh law:

we obtain 2 numbers x_i and z_i from the normal distribution with $\sigma = 1$ and $M = 0$;

we find the number $r_i = \sqrt{x_i^2 + z_i^2}$. (20)

This number will correspond to the number from the Rayleigh distribution with the parameter $\sigma = 1$.

Analogously, we can derive the quantities from the Rice distribution

$$P(r) = \frac{r}{\sigma^2} e^{-\frac{r^2+\alpha^2}{2\sigma^2}} I_0\left(\frac{\alpha r}{\sigma^2}\right), \tag{21}$$

where $I_0\left(\frac{\alpha r}{\sigma^2}\right)$ is a Bessel function;

α and σ are distribution parameters.

Taking into consideration that this law describes the probability distribution of the modulus of a random vector on a plane with independent components which have the dispersion σ and a mathematical expectation different from zero, we will use Formula (20) in this case as well, but as x_i we will take a number from a normal distribution with $\sigma = 1$ and $M = \alpha \neq 0$, since in this case we obtain the magnitude of the modulus of the random

vector for which the projections are independent and distributed according to the normal law with $M \neq 0$.

If we require numbers from the Simpson distribution, these can be derived by summing two numbers from a uniform distribution, since we know that the sum of two uniformly distributed quantities is distributed according to the Simpson law.

Numbers from the χ^2 distribution with one degree of freedom can be obtained by squaring the numbers taken from a normal distribution with mathematical expectation 0 and dispersion 1 (squares of normally distributed quantities with mathematical expectation 0 and dispersion 1 have distribution χ^2 with one degree of freedom). The derivation of the moduli of normally distributed quantities, of a logarithmically normal distribution and of other quantities presents no difficulties.

In conclusion of this section we consider a method for the derivation of random numbers distributed according to the Poisson law

$$P(k) = \frac{a^k}{k!} e^{-a} \tag{22}$$

with mathematical expectation a.

We use the Poisson limit theorem: if P_n is the probability of event A in a single test, the probability of the occurrence of k events in n independent tests as $n \to \infty$ and $P_n \to 0$ is asymptotically equal to $P(k)$.

We will model series of n independent tests in which the occurrence probability for events A is given by P_n which is on the order of 0.1–0.2. In this case, the following condition will be satisfied between a, P_n and n:

$$P_n = \frac{a}{n}, \tag{23}$$

which can be achieved by appropriate selection of the test number n. For the numbers x_i distributed according to the Poisson law we should then select the number of cases of the actual occurrence of event A.

C. Obtaining the Realizations of Random Functions

In a number of cases it becomes necessary to obtain the reallizations of random functions. As an example of such realizations we can cite the temperature of air, the speed and direction of wind as a function of altitude, engine thrust as a function of the time of engine operation, the fuel consumption in the fuel tank as a function of the distance traveled, etc. It is impossible to reduce the problem to the determination of random quantities at separate points, since a correlation link – these quantities are dependent – exists between the quantities at the individual points.

Let us stress that random functions must be distinguished from the nonrandom functions of random quantities (for example, the nonrandom function of aircraft flight range relative to the weight of the charged fuel which is a random quantity). The derivation of nonrandom functions of a random quantity presents no difficulties (with one of the methods described above, we obtain a random quantity and calculate its function).

The function whose ordinates for any fixed values of the argument are random quantities is referred to as a random function $x(t)$ of the argument t.

The basic characteristics of the random function are its mathematical expectation $m_x(t)$ and the dispersion $D_x(t)$, nonrandom functions of the argument, and the normed correlation function $r_x(t_1, t_2)$, a nonrandom function of two variable values of the argument at two points. All these characteristics are determined by processing experimental data by the conventional methods of mathematical statistics.

For any kind of transformation, as well as for utilization in models, a random function is conveniently presented in the form of canonical expansions, the method for whose derivation was developed by V.S. Pugachev [64]. The random function is written in the following form:

$$X_A(t) = m(t) + \sum_{i=1}^{n} x_i f_i(t). \qquad (24)$$

Here x_i are independent random quantities with a mathematical expectation equal to zero;

$f_i(t)$ are nonrandom functions of the argument, which are also known as coordinate functions.

Without dwelling on the derivation of the formulas for the calculation of $D(x_i)$ and $f_i(t)$, we will present only the final results, taken from Reference [25].

Let the following be determined by experiment: the mathematical expectation, the dispersion $D_x(t)$ and the correlation function $K_x(t_1; t_2) = r_x(t_1; t_2)\sqrt{D_x(t_1) D_x(t_2)}$ of the random function $X_A(t)$. We denote

$$X(t) = X_A(t) - m_x(t) = \sum_{i=1}^{n} x_i f_i(t). \qquad (25)$$

Calculate the dispersions $x_i - D(x_i)$ and $f_i(t)$. We can use the following recurrence formulas for the calculation of $f_i(t_j)$ and $D(x_i)$:

$$f_i(t_j) = \frac{K_x(t_i; t_j) - \sum_{k=1}^{i-1} D_k f_k(t_i) f_k(t_j)}{D_i}. \qquad (26)$$

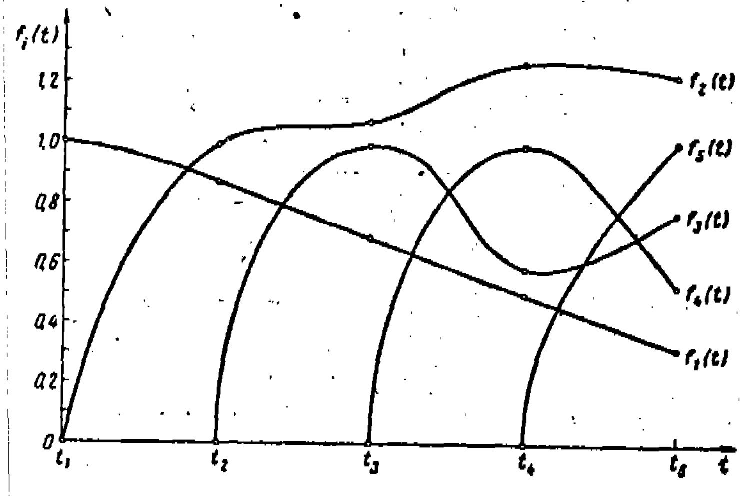

Fig. 2.2.1

It should be borne in mind in this case that when $i > j$

$$f_i(t_j) = 0, \tag{27}$$

and $f_i(t_i) = 1$, i.e., the coordinate functions have the form shown in Fig. 2.2.1.

EXAMPLE. Present the random function given by the following matrix of correlation moments in canonical form (i.e., given by the values of the correlation function).

TABLE 2.2.1

t_i \ t_j	1	2	3	4	5
1	0.16	0.14	0.11	0.08	0.05
2	—	0.20	0.18	0.17	0.14
3	—	—	0.23	0.20	0.19
4	—	—	—	0.26	0.22
5	—	—	—	—	0.28

The sequence of calculation is the following (we limit ourselves to two values):

$$D_1 = K_x(t_1;\ t_1) = 0.16,$$
$$f_1(t_1) = 1.0,$$
$$f_1(t_2) = \frac{K_x(t_1;\ t_2)}{D_1} = \frac{0.14}{0.16} = 0.87,$$
$$f_1(t_3) = \frac{K_x(t_1;\ t_3)}{D_1} = \frac{0.11}{0.16} = 0.69,$$
$$f_1(t_4) = \frac{K_x(t_1;\ t_4)}{D_1} = \frac{0.08}{0.16} = 0.50,$$
$$f_1(t_5) = \frac{K_x(t_1;\ t_5)}{D_1} = \frac{0.05}{0.16} = 0.31;$$

$$D_2 = K_x(t_2; t_2) - [f_1(t_2)]^2 D_1 = 0,20 - 0,87^2 \cdot 0,16 = 0,08,$$

$$f_2(t_1) = 0,$$

$$f_2(t_2) = 1,0,$$

$$f_2(t_3) = \frac{K_x(t_2; t_3) - D_1 f_1(t_2) f_1(t_3)}{D_2} = \frac{0,18 - 0,16 \cdot 0,87 \cdot 0,69}{0,08} = 1,05,$$

$$f_2(t_4) = \frac{K_x(t_2; t_4) - D_1 f_1(t_2) f_1(t_4)}{D_2} = \frac{0,17 - 0,16 \cdot 0,87 \cdot 0,50}{0,08} = 1,25,$$

$$f_2(t_5) = \frac{K_x(t_2; t_5) - D_1 f_1(t_2) f_1(t_4)}{D_2} = \frac{0,14 - 0,16 \cdot 0,87 \cdot 0,31}{0,08} = 1,21;$$

$$D_3 = K_x(t_3; t_3) - \{[f_1(t_3)]^2 D_1 + [f_2(t_3)]^2 D_2\} = = 0,23 - 0,69^2 \cdot 0,16 + 1,05^2 \cdot 0,08 = 0,07.$$

$$f_3(t_1) = 0,$$

$$f_3(t_2) = 0,$$

$$f_3(t_3) = 1,0,$$

$$f_3(t_4) = \frac{K_x(t_3; t_4) - [D_1 f_1(t_3) f_1(t_4) + D_2 f_2(t_3) f_2(t_4)]}{D_3} = = \frac{0,20 - 0,16 \cdot 0,69 \cdot 0,50 + 0,08 \cdot 1,05 \cdot 1,25}{0,07} = 0,57,$$

$$f_3(t_5) = \frac{K_x(t_3; t_5) - [D_1 f_1(t_3) f_1(t_5) + D_2 f_2(t_3) f_2(t_5)]}{D_3} = = \frac{0,19 - 0,16 \cdot 0,69 \cdot 0,31 + 0,08 \cdot 1,05 \cdot 1,21}{0,07} = 0,77;$$

$$D_4 = K_x(t_4; t_4) - \{[f_1(t_4)]^2 D_1 + [f_2(t_4)]^2 D_2 + [f_3(t_4)]^2 D_3\} = = 0,26 \cdot 0,50^2 \cdot 0,16 + 1,25^2 \cdot 0,08 + 0,57^2 \cdot 0,07 = 0,07.$$

$$f_4(t_1) = f_4(t_2) = f_4(t_3) = 0,$$

$$f_4(t_5) = 1,0,$$

$$f_4(t_5) = \frac{K_x(t_4; t_5) - [D_1 f_1(t_4) f_1(t_5) + D_2 f_2(t_4) f_2(t_5) + D_3 f_3(t_4) f_3(t_5)]}{D_4} = = \frac{0,22 - 0,16 \cdot 0,50 \cdot 0,31 + 0,08 \cdot 1,25 \cdot 1,21 + 0,07 \cdot 0,57 \cdot 0,79}{0,07} = 0,55;$$

$$D_5 = K_x(t_5; t_5) - \{[f_1(t_5)]^2 D_1 + [f_2(t_5)]^2 D_2 + [f_3(t_5)]^2 D_3 + + [f_4(t_5)]^2 D_4\} = 0,28 - 0,31^2 \cdot 0,16 + 1,21^2 \cdot 0,08 + 0,77^2 \cdot 0,08 + + 0,55^2 \cdot 0,07 = 0,09,$$

$$f_5(t_1) = f_5(t_2) = f_5(t_3) = f_5(t_4) = 0,$$

$$f_5(t_5) = 1,0.$$

The coordinate functions calculated in this example are shown in Fig. 2.2.1.

If a canonical expansion of the random function has been obtained, there will be no difficulty in obtaining its random realization. Indeed, in this case it is enough to have n random numbers x_i exhibiting the dispersions D_i and the required distribution, which is accomplished in the manner demonstrated in the previous section, followed by the calculation

$$
\begin{aligned}
x(t_1) &= m_x(t_1) + x_1 f_1(t_1), \\
x(t_2) &= m_x(t_2) + x_1 f_1(t_2) + x_2 f_2(t_2), \\
x(t_3) &= m_x(t_3) + \sum_{i=1}^{3} x_i f_i(t_3), \\
&\cdots\cdots\cdots\cdots\cdots \\
x(t_n) &= m_x(t_n) + \sum_{i=1}^{n} x_i f_i(t_n).
\end{aligned} \tag{28}
$$

This method of deriving random functions is most convenient for a computer, although not the only method.

We frequently encounter the so-called *stationary random functions* for which

$$m_x(t) = \text{const}, \tag{29}$$

$$K_x(t_1; t_2) = K_x(\tau), \tag{30}$$

where $\tau = t_2 - t_1$, i.e., with stationary functions the correlation function is independent of the magnitude of the arguments, but depends exclusively on their difference. The dispersion of the stationary function is constant.

For stationary random functions, rather than the canonical expansion, it is simpler to obtain the so-called spectral expansion, i.e., the representation of the expansion in the form of the sum of harmonic oscillations exhibiting various amplitudes and frequencies. This expansion may be interpreted as a special case of the canonical expansion in which trigonometric functions play the role of coordinate functions. In the interval $-T + T$

$$X(t) = \sum_{i=0}^{n} \left(x_{1,i} \cos \frac{\pi}{T} it + x_{2,i} \sin \frac{\pi}{T} it \right), \tag{31}$$

where n is a sufficiently large number,

$x_{1,i}$ and $x_{2,i}$ are random numbers, the dispersions for each pair of which are equal to one another and on the basis of the familiar correlation function are determined from the following formulas:

$$D_0 = \frac{1}{T} \int_0^T K_x(\tau)\, d\tau, \tag{32}$$

$$D_i = \frac{2}{T} \int_0^T K_x(\tau) \cos \frac{\pi}{T_i} \tau d\tau. \tag{33}$$

With the spectral expansion of the random function, it is rather simple to achieve its random realization, using the random numbers and Formula (31).

D. Formation of Random Flows

First of all, let us note that in the case of complex flows it is best to introduce experimental data into the model. Here we

will examine the formation only of the simplest flow and we refer the readers to References [12, 13] in which are discussed methods for the formation of random flows with uniformly distributed intervals between calls, methods for the formation of Erlang [sic] flows, generalizations of Erlang flows, flows with fixed minimum time, flows with variable parameters, as well as for flows of more general character.

Thus, given the requirement for the formation of the simplest flow, i.e., determine the times t_k for the arrival of requisitions. Let us represent these times in the following form:

$$t_1 = \xi_1,\quad t_2 = \xi_1 + \xi_2,\quad \ldots\ldots,\quad t_k = \sum_{i=1}^{k} \xi_i. \tag{34}$$

The function for the density of any intervals between calls ξ_i for the simplest flow has the form

$$f(z) = \lambda e^{-\lambda z}. \tag{35}$$

Therefore, the construction of the realization of a flow of simplest events can be reduced to the formation of a sequence of independent random numbers distributed exponentially. The method for the derivation of such numbers was covered earlier.

EXAMPLE. Formulate the simplest flow with $\lambda = 1$.

From the table of random numbers we obtain the numbers λ_i distributed according to the law of equal probability from 0 to 1; we will transform these into numbers distributed exponentially by means of Formula (10) and we will then determine the times of event occurrence by means of Formulas (34). The calculational results are presented in Table 2.2.2.

TABLE 2.2.2

n	1	2	3	4	5	6
λ_i	0.66674	0.99279	0.24202	0.94010	0.60981	0.13094
$\xi_i = -\frac{1}{\lambda}\ln(1-\lambda_i)$	1.110	3.933	0.277	2.814	0.939	0.141
$t_n = \sum_{i=1}^{n} \xi_i$	1.110	5.044	5.321	8.135	9.074	9.215

E. Modeling of Random Events

In statistical tests it frequently becomes necessary to answer the question whether or not a random event occurred, if its probability is known. For example, we consider a battle between two tank groups. A salvo has been fired. The probability of tank damage is known. The question that has to be answered: was the tank damaged or not.

First of all, let us consider the case in which this event is independent of all others. In this case, we have to select a number λ_i from the population of numbers uniformly distributed from 0 to 1 and compare that number with the probability P of the subject event:

If $P \geq \lambda_i$, event A occurred;

If $P < \lambda_i$, event A did not occur.

It is easy to prove that with a large number of tests the frequency of event A determined in this manner coincides with its probability [13].

With this method it is possible to form more complex independent events. As an example, let us consider the modeling of two dependent random events: A having the probability P_A and B having the probability P_B. Moreover, we know the conditional probability $P(B/A)$ of event B for the condition that event A has occurred.

We derive λ_i as the random number from a population uniformly distributed in the interval 0-1. If $\lambda_i < P_A$, we assume that event A has occurred. In this case, for tests with event B we employ the conditional probability $P(B/A)$. We obtain λ_{i+1} and if $\lambda_{i+1} < P(B/A)$, we assume that event B has occurred. In the opposite case, we assume that event B has not occurred.

If $\lambda_i > P_A$, event A has not occurred. In this case, for a test associated with event B, we must use the probability

$$P(B/\bar{A}) = \frac{P_B - P_A P(B/A)}{1 - P_A}, \tag{36}$$

i.e., the probability of event B for the condition that event A has not occurred. At the conclusion of this part we will recall the relationship of the correlation coefficient forevents A and B [$r(AB)$] with the events probabilities P_A and P_B and the conditional probability $P(B/A)$:

$$r(AB) = \frac{P_A P(B/A) - P_A P_B}{\sqrt{P_A(1-P_A)P_B(1-P_B)}}. \tag{37}$$

EXAMPLE. The probability of the first and second artillery shells fired at a target from an automatic canon is equal to 0.6,

i.e., $P_A = 0.6$ and $P_B = 0.6$. However, these events (the shells hitting the target under conditions of firing from an automatic canon) are dependent in view of the presence of common firing errors. The probability of the second shell hitting the target under the condition that the first shell has hit is therefore higher than 0.6. Let it be equal to 0.8 [$P(B/A) = 0.8$]. We have to model these random events.

Using the table of random numbers uniformly distributed in the interval 0-99999 (Table 2 of the appendix), we find the random number 57705. This will correspond to a number from those uniformly distributed in the interval 0-1

$$\lambda_1 = \frac{57705}{99999} = 0.57705.$$

$\lambda_1 < 0.6$; consequently, event A (the first shell hitting) has occurred. In this case, to determine whether or not event B occurred, we use $P(B/A) = 0.8$.

We determine

$$\lambda_2 = \frac{71618}{99999} = 0.71618 < 0.8.$$

Consequently, the second hit also occurred.

If the relationship between the events were not taken into consideration, $\lambda_2 > P(B) = 0.6$ and we would have to assume that the event did not occur.

§2.3. EVALUATION OF ACCURACY IN RESULTS DERIVED BY THE METHOD OF STATISTICAL TESTS

The problem of the accuracy of the results derived by the method of statistical tests is a fundamental question in the method, since essentially it governs the applicability of the method. Indeed, if the method of statistical tests is to be used to select a solution from among many, and for the evaluation of each of these many solutions with a sufficiently high degree of accuracy we have to derive a large number of realizations, it may turn out that the expenditure of machine time will be so great that we will have to reject the application of the method.

A. General Problems in the Evaluation of Method Accuracy

There are three basic reasons leading to errors in the application of the method of statistical tests:

inaccuracy in the determination of the input data;

methodological errors associated with simplifications of the model and failure by the model to make allowance for certain factors;

errors associated with smallness of the selection and, moreover, calculation errors which, as a rule, can be neglected.

The accuracy of the method can be described by the following formula:

$$\sigma_{\text{cи}} = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial K}{\partial \mu_i}\right)^2 \sigma_{\mu_i}^2 + \sigma_{\text{м}}^2 + \sigma^2}, \qquad (1)$$

where σ_{si} is the total root mean square error of the method;

K is the criterion determined at the conclusion of the application of the method (the mathematical expectation or the root mean square deviation of some quantity, the probability of some event);

μ_i represents the factors affecting the magnitude of the criterion;

σ_{μ_i} is the root mean square error in the determination of the magnitudes of these factors;

σ_m is the methodological error. The determination of this error when, for simplification of the model, after investigation of the effects of various factors we neglect the secondary factors, is not difficult

$$\sigma_{\text{м}} = \sqrt{\sum_{k=1}^{n_1} \left(\frac{\partial K}{\partial \mu_k} \sigma_{\mu_k}\right)^2}, \qquad (2)$$

where μ_k represents the magnitudes of the factors whose effect we have neglected. However, if this error, arising as a result of impossibility of describing the effect of any factors by analytical relationships, such a model must be checked out by other methods (testing of the physical model, evaluation at limit assumptions, etc.) in order to arrive at a conception of the magnitude of σ_m.

σ is an error associated with the small number of realizations. In the following sections of this chapter we will examine the magnitude of this error in detail.

It should be borne in mind that a change in the magnitudes of the components in the total error in those cases in which they are markedly smaller than the remaining does not lead to a significant change in the total error. Therefore, if any of the components is sufficiently small (on the order of 0.5 of the largest), its reduction leads to no noticeable lowering of the total error.

If the determining error is the error due to inaccuracies in the determination of input data, the permissible error from limitation on the number of realizations can be determined from the formula

$$\sigma = 0{,}5 \sqrt{\sum_{i=1}^{n} \left(\frac{\partial K}{\partial \mu_i} \sigma_{\mu_i}\right)^2}. \qquad (3)$$

If σ is indeed smaller, the number of realizations can be reduced.

In constructing a model we should also analyze such problems as the possibility of model simplification. This increases the methodological error, but reduces the time spent on achieving one realization and at the same time for a given machine time makes it possible to increase the number of realizations and, consequently, to reduce the error resulting from the smallness of that number. Thus it is possible to find the optimum complexity of the model assuring the minimum magnitude of the total error for the given machine time.

Below we examine the errors associated with a small number of realizations.

B. Accuracy of Determining the Mathematical Expectation of the Sought Quantity

As a result of N statistical tests we have obtained the following values for the sought quantity: x_1, x_2, x_3, ..., x_N and we have calculated

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \tag{4}$$

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - x)^2. \tag{5}$$

The quantity x is taken as the mathematical expectation of the sought quantity.

In mathematical statistics (see [71], page 215) the following is demonstrated

$$\alpha = \text{Prob}\,(-\varepsilon \leqslant \bar{x} - x_0 \leqslant \varepsilon), \tag{6}$$

where x_0 is the true value for the mathematical expectation of the sought quantity;

$$\varepsilon = t_\alpha \frac{S}{\sqrt{N}}, \tag{7}$$

t_α is the α function and $k = N - 1$ for the Student distribution whose values are given in Table 4 of the appendix.

In order for us todetermine the mathematical expectation of the criterion such that the error does not exceed a given value for a given degree of reliability, we have to use Formula (7) and the table. For convenience in introducing the tables in the appendix into the computer, approximation polynomials are offered.

The root mean square deviation of the quantity $\bar{x} - x_0$ according to [71] is

$$\sigma(\bar{x} - x_0) = \frac{S}{\sqrt{N}}\sqrt{\frac{N-1}{N-3}}. \tag{8}$$

This formula should be used when we have to calculate total errors, for example, by means of Formula (1).

The cited formulas make it possible to evaluate the accuracy of the derived results or to evaluate the adequacy of the number of realizations.

EXAMPLE. Determine the mathematical expectation of a quantity calculated by the method of statistical tests if it is required that the error not exceed 7% of the mathematical expectation with a probability of 0.9.

It is most expedient here, after derivation of each realization N, to calculate $\bar{x}$ and S according to Formulas (4) and (5); then, from $\alpha = 0.9$ and $k = N - 1$ in Table 4 of the appendix to find t_α and by means of Formula (7) to calculate the ε which is then referred to $\bar{x}$. If this quantity is equal to 7% or less, the calculation can be curtailed. Otherwise, it must be continued.

The indicated method is carried out well on computers.

Table 2.3.1 shows the results of calculations by the method described. For the number x_i we have taken the random sequence of numbers from Table 1 of the appendix, beginning with the 21st number, increased by 5.

TABLE 2.3.1

N	x_i	$\bar{x}$	S	t_α	ε	$\frac{\varepsilon}{\bar{x}}$	σ
1	4,33	4,33	—	—	—	—	—
2	5,61	4,97	0,64	—	—	—	—
3	6,15	5,36	0,72	2,920	1,21	0,23	—
4	4,91	5,25	0,62	2,353	0,73	0,14	0,54
5	4,10	5,02	0,71	2,132	0,67	0,13	0,45
6	4,30	4,90	0,69	2,015	0,56	0,12	0,36
7	4,64	4,86	0,63	1,943	0,47	0,10	0,29
8	5,05	4,89	0,59	1,902	0,40	0,08	0,23
9	5,56	4,96	0,59	1,860	0,37	0,08	0,23
10	6,28	5,09	0,68	1,836	0,40	0,08	0,25
11	3,82	4,98	0,74	1,812	0,41	0,08	0,25
12	4,34	4,92	0,73	1,797	0,38	0,08	0,23
13	4,32	4,88	0,71	1,782	0,36	0,07	0,22

A limitation may also be set with respect to σ, in which case it has to be calculated by means of Formula (8) and compared with the given.

C. Accuracy of Determining the Sought Root Mean Square Deviation

As an appropriate value for the root mean square deviation we assume S, calculated with Formula (5).

In mathematical statistics([71], page 217) the following is demonstrated:

$$\alpha = \text{Prob}\,[S \leqslant q\sigma], \tag{9}$$

where σ is the true value of the root mean square deviation;
q is determined by means of Table 5 of the appendix on the basis of α and $k = N - 1$.

Reference [82] presents an approximate formula for $\sigma(S)$ which is in good agreement with the exact formula

$$\sigma(S) \doteq \frac{\sigma}{\sqrt{2N-1.4}}. \tag{10}$$

Formula (9) is used for the case in which we have to determine σ with an error not to exceed the given value. In this case, the required number of realizations can be determined prior to the start of the tests.

Formula (10) is used in those cases in which we have to determine the total error. Its application is made difficult by the fact that the quantity σ (general dispersion), as a rule, is unknown. With a sufficiently large number of tests in this case instead of σ we must assume approximately S.

EXAMPLE. Find the number of realizations for the determination of σ if we require, with probability 0.95, that it not exceed the true value by more than 20%.

By means of Table 5 of the appendix, having taken $q = (1/1.2) = 0.834$ and $\alpha = 1 - 0.95 = 0.05$, we find $K = 50$, whence

$$N = K + 1 = 50 + 1 = 51.$$

D. Accuracy of Determining the Probability of Some Event

Let there have been carried out N tests in which the subject event occurred m times. The probability of this event occurring will be defined as

$$P = \frac{m}{N}. \tag{11}$$

The root mean square deviation for P from the true value (P_0) can be determined by means of the following formula:

$$\sigma(P) = \sqrt{\frac{P_0(1-P_0)}{N}}. \tag{12}$$

This formula is useful in evaluating the total error of the method; however, not knowing the exact magnitude of P_0 makes the utilization of this formula more difficult. Occasionally, instead of P_0 we have to assume P approximately.

In determining the probability P by the method of statistical tests, it is frequently necessary to determine the probability α that P will not differ from the true value (P_0) by more than a definite quantity ΔP, or by a quantity $\Delta P = |P - P_0|$ which will not be exceeded, with the given probability α. The quantity ΔP is known as the *confidence interval*.

We have to use Table 6 of the appendix to determine this quantity. In N statistical tests, if the subject event has not occurred a single time, we find R_0 from Table 6, and from this value we calculate

$$P_v = \frac{R_0}{N}. \tag{13}$$

The quantity P_v is the upper confidence boundary and the lower confidence boundary in this case is $P_n = 0$.

In N statistical tests, if the subject event has occurred m times, according to α, m/N and m we find the coefficients R_1 and R_2 in Table 6, and by means of these we calculate the confidence boundaries

$$P_n = \frac{m}{NR_1}, \tag{14}$$

$$P_v = \frac{m}{NR_2}. \tag{15}$$

We should bear in mind that if

$$\text{Prob}\,(P \geqslant P_n) = \alpha_1, \tag{16}$$

$$\text{Prob}\,(P \leqslant P_v) = \alpha_2, \tag{17}$$

then

$$\text{Prob}\,(P_n \leqslant P \leqslant P_v) = \alpha_1 + \alpha_2 - 1. \tag{18}$$

EXAMPLE. a) During the course of the statistical tests we have to determine the probability of an event with an error not exceeding 0.20, with a reliability of 0.90. For 20 tests we have $m = 3$.

The question arises as to whether the derivation of the realizations should be continued or whether they can be stopped.

By means of Table 6, having taken $m/N = 0.15$, $m = 3$ and $\alpha = 0.95$ [see Formulas (16), (17) and (18)], we find

$$R_1 = 3{,}56,\ R_2 = 0{,}43,$$

$$P_n = \frac{m}{NR_1} = 0{,}04, \quad P_v = 0{,}35,$$

$$\text{Prob}\,(0{,}04 \leqslant P \leqslant 0{,}35) = 0{,}90,$$

$\Delta P_1 = 0{,}35 - 0{,}15 = 0{,}20$, $\Delta P_2 = 0{,}15 - 0{,}04 = 0{,}11$, i.e., do not exceed 0.20. Consequently, the derivation of the realizations can be stopped. We note that $\Delta P_1 > \Delta P_2$ always, and we can therefore limit ourselves to determination of ΔP_1.

b) For $N = 20$ let $m = 0$: given $\alpha = 0.975$ and a confidence interval no higher than 0.20.

From Table 6 we find $R_0 = 3.37$. Then $P_v = (R_0/N) = 0.17$. $\Delta P = 0.17 - 0.00 = 0.17 < 0.20$. The calculation can be stopped.

E. Accuracy in Determining the Function of a Single Variable in the Given Interval

Frequently, in solving operations research problems it becomes necessary to seek not one constant, but a function on the given interval of argument variation. Let us, first of all, consider the problem in which the form of this function is known, in particular, let this be a polynomial of degree n. In this case, we have to determine the average magnitude of the root mean square error σ_{sr} of the function on a given interval and to select the method of determining that function so that for a given number of tests we obtain the minimum σ_{sr} to determine the function. Let the given interval be 0-τ.

Let us consider the simplest case in which the sought function is a first-degree polynomial

$$y = F(x) = ax + B \tag{19}$$

in the 0-τ interval.

To determine the coefficients of this polynomial we have to calculate the values of $F(x)$ at least at two points $(x_1; y_1)$ and $(x_2; y_2)$. Then

$$y = y_1 \frac{x_2 - x}{x_2 - x_1} + y_2 \frac{x - x_1}{x_2 - x_1}. \tag{20}$$

The quantities x_1 and x_2 contain virtually no errors, and y_1 and y_2 are determined experimentally with errors characterized by σ_{y_1} and σ_{y_2}, a result of the limitations on the number of tests.

On the basis of the theorem of linear function dispersion, we can write

$$\sigma_y^2 = \left(\frac{x_2 - x}{x_2 - x_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{x - x_1}{x_2 - x_1}\right)^2 \sigma_{y_2}^2.$$

Then

$$\sigma_{\text{cp}_1}^2 = \frac{1}{\tau}\int_0^\tau \left[\left(\frac{x_2 - x}{x_2 - x_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{x - x_1}{x_2 - x_1}\right)^2 \sigma_{y_2}^2\right] dx =$$

$$= \frac{\tau^2(\sigma_{y_1}^2 + \sigma_{y_2}^2) - 3\tau(x_2\sigma_{y_1}^2 + x_1\sigma_{y_2}^2) + 3(x_2^2\sigma_{y_1}^2 + x_1^2\sigma_{y_2}^2)}{3(x_2 - x_1)^2}. \tag{21}$$

Let us now seek the optimum quantities x_1 and x_2 at which $\sigma_{sr_1}^2$ is minimum. Considering that $\sigma_{sr_1}^2$ is symmetric with respect to $\tau/2$, we can conclude that the optimum variant must be the one for which

$$\sigma_{y_1} = \sigma_{y_2} = \sigma_y \text{ and } x_2 = \tau - x_1.$$

Then

$$\sigma^2_{cp_1} = \frac{2\tau^2\sigma_y^2 - 6\tau\sigma_y^2 x_1 + 3\sigma_y^2 x_1^2}{3(\tau - 2x_1)^2}. \qquad (22)$$

From the condition $\frac{\partial \sigma^2_{cp_1}}{\partial x_1} = 0$ we find $x_1 = 0$, i.e., the points x_1, y_1 and x_2, y_2 should be set at the ends of the interval.

In this case the calculations show that $\sigma_{cp_1} = 0{,}817\sigma_y$. Analogous studies were carried out for polynomials of higher degrees. It was assumed in this case that the points x_1, y_1; x_2, y_2; ... are situated uniformly over the given interval, beginning from the ends, and that their number is minimum, i.e., equal to the power of the polynomial plus unity. Studies show that the minimum number of points assures a minimum σ_{sr_1}. Analysis of the formulas for σ_{sr_1} makes it possible to select the optimum distribution of the number of tests over the points. If we assume that $\sigma_{y_i} = \frac{\sigma}{\sqrt{n_i}}$, where n_i is the number of tests at the given point, and if we impose the condition $\sum_{i=1}^{n+1} n_i = N$, we can determine the optimum n_i ensuring the minimum σ_{sr_1}. Corresponding data are presented in Table 2.3.2. That table also shows the coefficient K_1 from the formula

$$\sigma_{cp_1} = \frac{\sigma}{\sqrt{\frac{N}{K_1}}}. \qquad (23)$$

In a number of cases σ is unknown and we have to assume S for it.

TABLE 2.3.2

Degree of polynomial n	Most advantageous number of tests at the points in sequence	Values of K_1: For an equal number of tests at each point	Values of K_1: For the most advantageous distribution of the number of tests over the points
0	1.000N	1.00	1.00
1	0.500N; 0.500N	1.34	1.34
2	0.250N; 0.500N; 0.250N	2.40	2.14
3	0.153N; 0.347N; 0.347N 0.153N	3.69	3.22
4	0.112N; 0.280N; 0.216N; 0.280N; 0.112N	5.33	4.71

However, in a number of cases the exponent of the sought function is unknown. In this case, in addition to the error due to the small volume of selection at the reference points, there may also arise an error due to the form of an improperly selected function. Let us consider the problem of this error. First of all, let us see what results from the selection of a first-degree polynomial instead of a second-degree polynomial under the condition that the mean magnitudes of all terms in the given interval $(0-\tau)$ are equal to each other.

In this case the true polynomial is written as follows:

$$y_{(2)} = \frac{\sigma_F}{\sqrt{11.133}} \left(\frac{3x^2}{\tau^2} + \frac{2x}{\tau} + 1 \right), \qquad (24)$$

where σ_F is the root mean square value of the sought function in the given interval.

If instead of this polynomial we take a first-degree polynomial for which the quantities y coincide with the corresponding values of y_2 from (24) at the ends of the interval, the polynomial will have the following form:

$$y_1 = \frac{\sigma_F}{\sqrt{11.133}} \left(\frac{5x}{\tau} + 1 \right). \qquad (25)$$

The root mean square error in the subject interval is determined in the following manner:

$$\sigma^2_{cp_2} = \int_0^1 [y_{(2)} - y_{(1)}]^2 \, dx = \frac{\sigma_F^2}{36}. \qquad (26)$$

It turns out that σ_{sr_2} is also a function of the combination of the signs of $y_{(2)}$. If we assume all combinations of signs to be equiprobable, analogous calculations for each combination of signs and the average σ_{sr_2} will give us

$$\sigma_{cp_2} = 0,375\sigma_F = K_2\sigma_F. \qquad (27)$$

Analogous calculations were carried out for other cases as well. Table 2.3.3 shows the coefficients K_2 where the degree of the true polynomial is equiprobable from 0 to M, and the degree of the adopted polynomial is n.

In practical applications it is difficult to expect that the sought function will exhibit a large number of extrema and inflexions, and for this reason it will, for all intents and purposes, always be well approximated by polynomials of the 3rd-4th degrees. In any event, we can expect that $M < 6$.

TABLE 2.3.3

n \ M	1	2	3	4	5	6	7
0	0,000	0,270	0,353	0,453	0,502	0,556	0,591
1	0,000	0,000	0,125	0,252	0,339	0,435	0,533
2	0,000	0,000	0,000	0,026	0,053	0,087	0,119
3	0.000	0.000	0,000	0,000	0,005	0,013	0,020

Let us now consider the problem of selecting the optimum degree of the approximating polynomial. The total mean error of the sought function in the given interval (taking into consideration that σ_{sr_1} and σ_{sr_2} are independent)

$$\sigma_{cp}=\sqrt{\sigma_{cp_1}^2+\sigma_{cp_2}^2}=\sqrt{\frac{\sigma^2K_1}{N}+K_2^2\sigma_F^2}, \tag{28}$$

$$\frac{\sigma_{cp}}{\sigma_F}=\sqrt{m^2K_1+K_2^2},$$

where

$$m=\frac{\sigma}{\sigma_F\sqrt{N}}. \tag{29}$$

Since K_1 and K_2 for fixed M are functions of n, σ_{sr}/σ_F will be a function of n and m.

The calculational results are shown in Fig. 2.3.1 from which we can see that there exists an optimum quantity n which is a function of m. The physical significance of m is the following: it is the relative accuracy in the determination of the function at the reference point. The higher this accuracy, the greater the degree of the polynomial that should be assumed. With $m > 0.2$ it is advisable to take $n = 2$; with $m < 0.2$, the quantity n is best taken as equal to 3.

EXAMPLE. Find the probability of target damage as a function of the ratio of the radius of the damage zone to the root mean square deviation $R = r/\sigma$ in the interval $R_1 = 1-3$; in this case,

$P=\int_0^{R_1} Re^{-\frac{R^2}{2}}dR$ is calculated by the method of statistical tests.

(This example was selected only for convenience in comparison with exact results, since in actual practice this function is calculated analytically). We can carry out 100 tests to calculate this function.

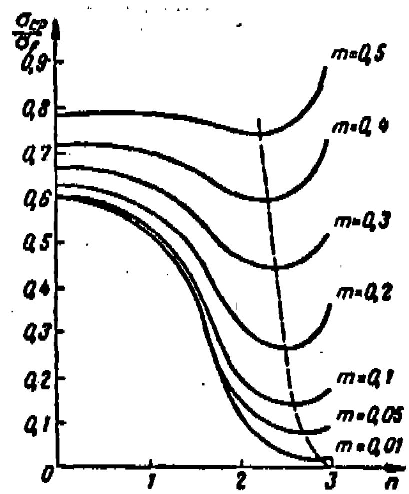

Fig. 2.3.1

Regardless of the degree of the polynomial, in the form in which we will seek the function we will have to carry out the tests at the ends of the interval. We can see from Table 2.3.2 that as a function of the degree of the polynomial, in which form we will seek the function, from 50 to 11.2% of the tests will have to be carried out at the ends of the interval. Let us carry out sets of 11 tests and determine σ.

For the point with $R_1 = 1$ we will use the calculations shown in Table 2.1.2, whence for $n = 11$, $S_{(1)} = 0.157$, $P_{(1)} = \mathcal{Y}_1 = 0.446$.

For the point with $R_1 = 3$ we will carry out analogous calculations as a result of which for $n = 11$ we obtain $S_{(3)} = 0.590$ and $P_{(3)} = 0.999$.

We now calculate σ (for σ we actually assume S):

$$\sigma \cong S = \sqrt{\frac{S_{(1)}^2 + S_{(3)}^2}{2}} = 0{,}55,$$

$$\sigma_F = \sqrt{\frac{P_{(3)}^2 + P_{(1)}^2}{2}} = 0{,}07,$$

$$m = \frac{\sigma}{\sigma_F \sqrt{N}} = \frac{0{,}55}{0{,}77\sqrt{100}} = 0{,}07.$$

Proceeding from m, we determine that the most advantageous degree of the polynomial will be 3.

The distribution of the number of realizations over the points must then be the following (see Table 2.3.2):

for $R_1 = 1$ — 15 realizations
$R_1 = 1{,}67$ — 35 realizations
$R_1 = 2{,}33$ — 35 realizations
$R_1 = 3$ — 15 realizations

Having carried out the corresponding calculations, we determine P_i and S_i.

TABLE 2.3.4

R_1	1	1,67	2,33	3
N_i	15	35	35	15
S_i	0,151	0,302	0,377	0,560
P_i	0,464	0,735	0,978	0,933

$$P = aR^3 + bR^2 + cR + d.$$

Having substituted the corresponding values of R_i and P_i into this formula and having solved the resulting system of equations for a, b, c, d, we will have the following formula for P:

$$P = -0{,}149R^3 + 0{,}711R^2 - 0{,}672R + 0{,}571. \tag{30}$$

Having calculated $\sigma_F=0.80$, $\sigma\approx0.375$, $m\cong0.06$, by means of Fig. 2.3.1 we will find that

$$\sigma_{cp}\cong 0.1\sigma_F = 0.08.$$

For comparison Fig. 2.3.2 shows the exact graph of the function and that obtained by means of Formula (30).

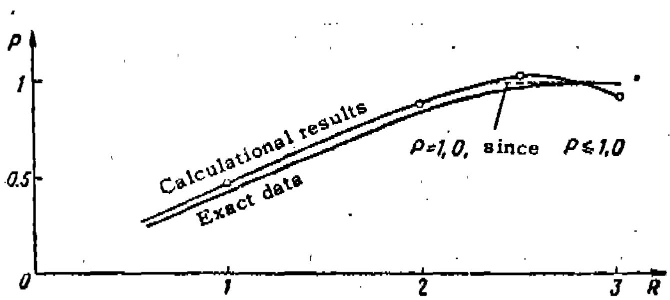

Fig. 2.3.2

F. Verification of the Hypothesis Pertaining to the Magnitude of the Mathematical Expectation

In operations research we very frequently encounter a case in which, of two variants (A and Б) we have to select one, whose mathematical expectation is greater than the criterion of mathematical expectation of the other. In this case, neither the mathematical expectation nor the dispersion of the criteria are known prior to the investigation, and the criteria themselves are determined by the method of statistical tests.

In this case, prior to the start of the tests it is impossible to answer the question of the number of the tests required. As the tests are carried out, it becomes necessary to calculate $\delta x_i = x_{A_i} - x_{Б_i}$, the mathematical expectation and the root mean square deviation of that quantity

$$\overline{\delta x}=\frac{1}{N}\sum_{i=1}^{N}\delta x_i,$$

$$S^2=\frac{1}{N}\sum_{i=1}^{N}(\delta x_i-\overline{\delta x})^2. \quad (31)$$

Then, assuming that for the determination of each of the subject quantities up to N have been carried out, we calculate the criterion

$$t=\frac{\overline{\delta x}}{S}\sqrt{N-1}. \quad (32)$$

Further, from Table 4 of the appendix, having taken $K = 2(N-1)$, we find the probability α that $\bar{x}_A > \bar{x}_Б$, i.e., that $\delta x > 0$. If

this probability corresponds to that which is to be obtained, the calculation is stopped. Otherwise, it is continued until the required probability is obtained.

EXAMPLE. Determine which mathematical expectation of the quantities x_A or $x_Б$ is the larger. The correct answer must be given with probability 0.95. The results of the calculations are shown in Table 2.3.5. The calculation can be stopped after 4 tests.

TABLE 2.3.5

N	x_A	$x_Б$	δx_i	$\overline{\delta x}$	S	t	K	α
1	0,33	—0,19	0,52	0,52	—	—	—	—
2	1,61	0,29	1,32	0,92	0,40	2,30	2	0,84
3	2,15	2,17	—0,02	0,61	0,53	1,64	4	0,82
4	0,81	—0,63	1,44	0,82	0,56	2,54	6	0,95
5	0,10	—0,11	0,21	0,69	0,54	2,55	8	0,96

G. Verification of the Hypothesis Pertaining to the Magnitude of the Root Mean Square Deviation

Occasionally it is necessary to select the variant (from A and B) for which the root mean square deviation of the criterion is larger than for the other. In this case we carry out N_1 tests for variant A, and on the basis of the results we calculate S_A and we carry out N_2 tests for variant B from which we calculate S_B. It is demonstrated in mathematical statistics ([71], page 245) that

$$\frac{S_A^2}{S_B^2} = F, \tag{33}$$

where F is the function $K_1 = N_1 - 1$ and $K_2 = N_2 - 1$ and the given probability that $S_A > S_B$.

Table 8 in the appendix gives the values of the function F for P = 90 and 98%. In that table we find selection for the case $N_1 = N_2$ (i.e., $K_1 = K_2$) and for the case of the optimum N_1 and N_2, by which we understand the combination of these such that for the given $N_1 + N_2$ the quantity P is maximum. We see from the table that the number of realizations for the variant with the smaller dispersion is best made as large as possible.

EXAMPLE. A series of 21 tests were carried out for each of the two variants, from which it was determined that

$$S_A = 10, \qquad S_B = 8.$$

We have to answer the question whether S_A is larger than S_B with the probability of a correct answer at 90%.

Determine whether the derivation of realizations should be

continued or whether it can be stopped. We have

$$K_A + K_B = 21 + 21 - 1 - 1 = 40,$$

$$\frac{S_A}{S_B} = \frac{10^2}{8^2} = 1.56.$$

We see from Table 8 of the appendix that $P < 90\%$, since $S_A^2/S_B^2 = 1.56 < 2.12$. Consequently, the derivation of realizations must be continued.

If we assume that with continuation of the experiments S_A/S_B will not change, we determine from the table that to achieve $P = 90\%$ the total number of experiments must be around 120.

In the given case it is necessary to carry out a successive analysis of the test results, increasing the number of tests for the variant with the smaller dispersion so that K_2 approaches the optimum. The latter, of course, yields a noticeable effect only with a small number of tests.

Introduction of the corresponding algorithm into the computer presents no difficulties.

H. Verification of Hypothesis Pertaining to the Magnitude of Probability

Finally, there is possible the case in which it is necessary to establish whether the probability corresponding to variant A is larger or smaller than the probability corresponding to variant B.

With a sufficiently large number of tests (on the order of 30) it may be assumed that

$$P_A - \frac{m_A}{N_A} \text{ and } P_B - \frac{m_B}{N_B}$$

will be distributed according to normal laws with the magnitudes of the root mean square deviations

$$\sigma_A = \sqrt{\frac{P_A(1-P_A)}{N_A}}; \qquad (34)$$

$$\sigma_B = \sqrt{\frac{P_B(1-P_B)}{N_B}}. \qquad (35)$$

In this case the differences $\frac{m_A}{N_A} - \frac{m_B}{N_B}$ will also be distributed according to the normal law with

$$\sigma = \sqrt{\sigma_A^2 + \sigma_B^2}. \qquad (36)$$

Then the probability α that $P_1 > P_2$ can be determined from the equation

$$\alpha = F\left[\frac{\frac{m_A}{N_A} - \frac{m_B}{N_B}}{\sqrt{\frac{P_A^2(1-P_A)^2}{N_A^2} + \frac{P_B^2(1-P_B)^2}{N_B^2}}}\right]. \tag{37}$$

If we assume $P_A = m_A/N_A$ and $P_B = m_B/N_B$, which can be done only with sufficiently large N_A and N_B, we have

$$\alpha = F\left[\frac{\frac{m_A}{N_A} - \frac{m_B}{N_B}}{\sqrt{\frac{m_A^2(N_A-m_A)^2}{N_A^4} + \frac{m_B^2(N_B-m_B)^2}{N_B^4}}}\right]. \tag{38}$$

Here :

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx. \tag{39}$$

The $F(x)$ tables are given in the appendix (see Table 3).

EXAMPLE. Thirty (30) tests have been carried out, and the event A occurred 15 times, and event B occurred 10 times. Check the hypothesis that probability of event A is greater than event B with reliability $\alpha = 0.975$.

$$\alpha = F\left[\frac{\frac{15}{30} - \frac{10}{30}}{\sqrt{\frac{15^2(30-15)^2}{30^4} + \frac{10^2(30-10)^2}{30^4}}}\right] = F(1{,}97) = 0{,}9756,$$

i.e., check the probability that $P_A > P_B$ is greater than 0.975, and that the derivation of the realizations can be stopped.

I. Verification of Hypothesis Pertaining to the Presence of Relationships Between Two Quantities

In investigating the method it may become necessary to establish whether or not a relationship exists between the criterion (x) and some other quantity (y), i.e., in other words, to answer the question whether this quantity (y) affects the criterion. For this we have to calculate the correlation factor by carrying out N tests to determine x and y:

$$r = \frac{\sum_{i=1}^{N}(x_i y_i - \bar{x}\,\bar{y})}{N S_x S_y}. \tag{40}$$

We then have to determine the parameter t:

$$t = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}}. \tag{41}$$

Having taken the quantities t and $K = N - 2$, by means of Table 4 of the appendix find the probability that the resulting correlation is not random.

EXAMPLE. In studying a model it was established as a result of 20 tests that between the criterion and one of the factors there exists the correlation factor $r = 0.4$. What is the probability that this relationship indeed exists?

$$t = \frac{0.4\sqrt{20-2}}{\sqrt{1-0.4^2}} = 1{,}23.$$

From Table 4 of the appendix for $t = 1.23$ and $K = 20 - 2 = 18$ we find $\alpha \leq 0.800$, i.e., the probability of the existence of a relationship is not sufficiently large for reliable affirmation of its existence.

In carrying out statistical tests the quantity α is generally given. Then, after derivation of each of the realizations it is necessary to calculate S_x and S_y; after this, from Formulas (40, 41) we calculate the quantities r and t and we use the table to calculate α.

This quantity α should be compared with the given and depending on the result of the comparison, the calculation should be stopped or continued. This algorithm is easily realized in a computer. To simplify the computer calculations, the tables of the corresponding functions give the approximate calculation polynomials, convenient for introduction into the computer.

In working practice we may encounter other cases of hypothesis verification, e.g., hypotheses pertaining to the nature of the distribution for the sought quantity. The description of these hypotheses is given in courses on mathematical statistics, for example, in Reference [71].

§2.4. MEANS OF REDUCING DISPERSION

Earlier we considered the accuracy of results in the fundamental cases which may be encountered in application of the method of statistical tests. From this consideration we see that this accuracy is a function of the number of tests carried out and the root mean square deviation is approximately proportional to the square root of that number. Thus if accuracy is to be increased by a factor of 10, the number of realizations must be increased by 100. This is by no means always possible. There therefore arises the question of reducing the dispersion of the measured quantities by other means. In this case the reduction of dispersion, in and of itself, cannot serve as the criterion of method feasibility. We must consider the magnitude of the dispersion for one and the same expenditure of time.

We presently know many such ways. They can all be divided into two groups:

1. A combination of statistical tests with analytical methods. This may be the most effective way. It represents a unique method of combating the "blindness" of the method.

2. Application of special selections. These methods have much in common with those employed for analogous purposes in

mathematical statistics.

Let us examine each of these procedures in greater detail, although we should bear in mind that in specific cases they may be used in conjunction with each other.

A. Combination of Statistical Tests and Analytical Methods

Let there be a requirement to determine the probability of hitting a target represented by a circle of radius r for given rocket and control system characteristics. The model for the solution of this problem consists of two basic blocks: the block to evaluate firing accuracy, at the output of which in each test we obtain the rocket deviations from the center of the target (r_i), and a block to evaluate the probability of hitting the target, this unit being built in various ways. It may be built on the principle of using the method of statistical tests. In this case, for each realization of the first block a comparison of the derived deviation r_i with the target radius r is carried out. If $r_i \leq r$, the target has been hit. Otherwise, we assume that the target has not been hit.

The probability (p) of hitting the target is defined as the ratio of the number (m) of hits to the total number of tests (n)

$$p=\frac{m}{n}. \tag{1}$$

This is a case in which the method of statistical tests is used in "pure" form.

Another principle is possible for the construction of the block to evaluate effectiveness. The data derived from the first block are subjected to statistical processing as a result of which the root mean square deviations with respect to range and direction (σ_x and σ_z) are calculated. Let these be equal to each other

$$\sigma_x=\sigma_z=\sigma. \tag{2}$$

We assume these to be equal to the selected root mean square S. In this case the probability of hitting the target can be determined from the formula

$$p=1-e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}=1-e^{-\frac{1}{2b^2}}, \tag{3}$$

where

$$b=\frac{\sigma}{r}. \tag{4}$$

Here we have a combination of the method of statistical tests with the analytical method. Let us compare the errors of these two cases.

According to the material covered in the previous section, the error in the determination of p in the first case after N realizations is

$$\sigma_{p_1} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{(1-e^{-\frac{2}{2b^2}})\,e^{-\frac{1}{2b^2}}}{N}}. \qquad (5)$$

To calculate σ_{p_2} we will use the approximate equality

$$\sigma_{p_2} = \left|\frac{\partial p}{\partial \sigma}\right| \sigma_s = \frac{1}{r}\cdot\frac{\partial p}{\partial b}\,\sigma_S. \qquad (6)$$

But

$$\sigma_S = \frac{\sigma}{\sqrt{2N-1{,}4}}, \qquad (7)$$

where

$$S = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})^2}\,, \qquad (8)$$

$$\frac{\partial p}{\partial b} = -\frac{e^{-\frac{1}{2b^2}}}{b^3}. \qquad (9)$$

Then

$$\sigma_{p_2} = \frac{e^{-\frac{1}{2b^2}}}{b^2\sqrt{2}\,\sqrt{N-0{,}7}}. \qquad (10)$$

Hence

$$\frac{\sigma_{p_1}}{\sigma_{p_2}} = \sqrt{\frac{N-0.7}{N}}\,b^2\sqrt{2\left(e^{\frac{1}{2b^2}}-1\right)}. \qquad (11)$$

Since p uniquely defines b, we can express $\sigma_{p_1}/\sigma_{p_2}$ as a function of p and N. The results of the corresponding calculations are shown in Fig. 2.4.1.

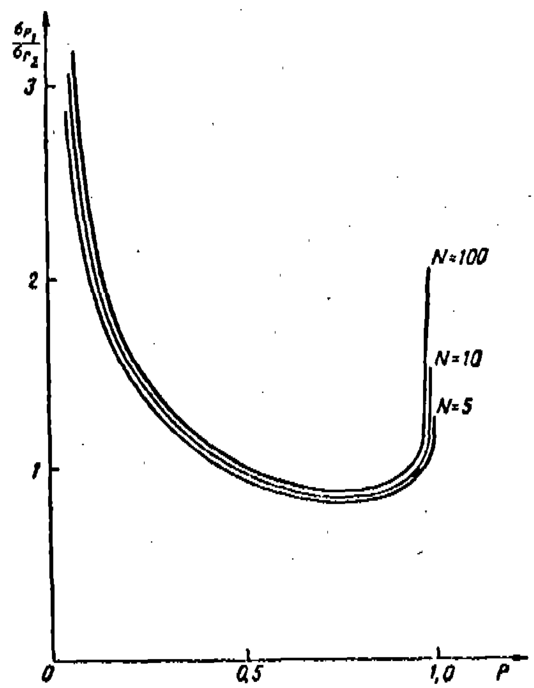

Fig. 2.4.1

We see from the figure that there exist regions where the first method is more advantageous, and regions where the second method is most advantageous. When $p = 0.05$ the second method leads to a root mean square deviation that is smaller by a factor of 3 than the second. And this actually indicates the possibility of reducing the number of realizations approximately by a factor of 10.

The cited example permits us to draw a conclusion as to the effectiveness of combining the method of statistical tests with analytical methods. The literature gives examples of considerably more effective combinations. It goes without saying that in certain cases it may prove more advantageous to investigate the first part of the model analytically, while the second part of the model is investigated by statistical tests.

B. Analysis of Results Obtained During the Realizations, and Adoption of Appropriate Changes in Calculation

In American literature this method is occasionally referred to as "Russian roulette." The essence of the method may be illustrated well by means of an example (see Table 2.1.2).

It is obvious that if $x_n > 1$ or $z_n > 1$, then $r_n > 1$ and $\delta m = 0$. Consequently, if it turns out that $x_n > 1$, the calculation need not be continued, but the assumption should be made immediately that the target has not been damaged.

If $x_n < 1$, we have to find z_n. If $z_n > 1$, it should be assumed that the target has not been damaged. Finally, if $x_n < 1$ and $z_n < 1$, it is advisable to carry out one more verification. Indeed, if $x_n \leq 0.7$ and $z_n \leq 0.7$, $r \leqslant \sqrt{0,7^2+0,7^2}=1$ and the target is damaged. This means that in this case r_n need not be calculated.

We see from Table 2.1.2 that according to the criterion $x_n >$ > 1 the calculation can be curtailed in 7 cases out of 20 and that according to the criterion $x_n < 1$, but with $z_n > 1$, the calculation can be stopped in 5 cases, while according to the criteria $x_n < 0.7$ and $z_n < 0.7$ the calculation can also be stopped in 5 cases. Thus in only 3 cases out of 20 is calculation of r_n necessary. Of course, in the subject example the calculation of r_n is not a difficult operation and the resulting savings in time is small. However, if the calculation of r_n and its comparison with r were the fundamental operations in terms of labor input (this occurs in the more complex calculation cases), the calculations could be carried out approximately 7 times more rapidly, which would make it possible, within the same period of time, to increase the accuracy of the derived results by a factor of approximately 2.5.

In statistical modeling of military action, when great losses

have been inflicted on one of the sides, it is obvious that there is no need to continue the calculation of this realization to its conclusion, but rather, it may be assumed that combat with that side has been lost. In this case, there arises the problem as to what is meant by "great losses." Most frequently the answer to this question can be given by analysis of the previous variants.

A clear example of the application of this method in statistical tests is given by analysis of the process of aerial reconnaissance with consideration of weather. If it turns out that the cloud cover screens the target, the calculation process may be stopped under the assumption that reconnaissance has produced no effect. In this case, the construction of the model must be such, first of all, as to provide the information required to carry out the evaluations, and then to carry out the difficult calculation.

Any complex model must be carefully analyzed and criteria for the cessation of calculation must be introduced in those cases in which intermediate results make it possible to adopt a given decision.

C. Determination of Parts Determined Analytically

We will clarify the essence of this method by several examples. First of all, when the integral is calculated with respect to the frequency of random points hitting an area bounded above by the integrated function, it is advisable to limit the subject area to the smallest possible magnitude, as follows from Fig. 2.4.2. It is evident that there is no need to use statistical tests to calculate that portion of the area AA'BB' which is easily calculated analytically.

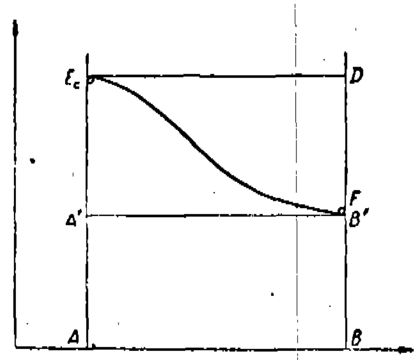

Fig. 2.4.2

As another example we can use the calculation $\mathcal{J}=\int_0^1 Re^{-\frac{R^2}{2}}dR$. From the integrand, let us isolate the linear portion and we will calculate

$$\mathcal{J}=\int_0^1 [f(R)+0,6R]\,dR=\int_0^1 f(R)\,dR+0,3, \tag{12}$$

$$f(R)=Re^{-\frac{R^2}{2}}-0,6R=R\left(e^{-\frac{R^2}{2}}-0,6\right). \tag{13}$$

Recalling that the exact value of this integral is equal to 0.393, we see that the essence of the method involves calculation of its basic part (0.3) analytically and only a small portion (0.093) by the method of statistical tests. The results of these calculations are shown in Table 2.4.1. The random numbers are the same as in Table 2.1.2.

From the table we see that the error $\mathcal{J}$ (1.5%) is considerably smaller than in the case shown in Table 2.1.2. The root mean square deviation (3.6%) is also considerably smaller here.

Let us examine yet another possible way of using the subject method. Let there be a requirement to calculate

$$p = \mathcal{J} = \int_0^3 Re^{-\frac{R^2}{2}} dR. \tag{14}$$

The results from the calculation of this quantity for n = 15 were presented in the previous section. The calculation accuracy can be considerably increased if we recall two circumstances:

1) $\int_0^\infty Re^{-\frac{R^2}{2}} dR = 1;$

2) rule 3σ, according to which the probability of deviation above 3σ is very small.

We will assume that the probability of deviation above 6σ is negligibly small.

TABLE 2.4.1

n	1	2	3	4	5	6	7	8	9	10
$\bar{J}$	0.134	0.011	0.090	0.040	0.140	0.051	0.120	0.137	0.137	0.132
$\mathcal{J}$	0.434	0.372	0.378	0.369	0.383	0.378	0.384	0.390	0.396	0.399
S	—	0.0949	0.0632	0.0548	0.0548	0.0490	0.0532	0.0548	0.0434	0.0527
σ	—	—	—	0.0475	0.0346	0.0258	0.0246	0.0230	0.0167	0.0189

n	11	12	13	14	15	16	17	18	19	20
$\bar{J}$	0.129	0.117	0.099	0.128	0.016	0.114	0.051	0.057	0.134	0.134
$\mathcal{J}$	0.402	0.403	0.403	0.405	0.399	0.400	0.397	0.395	0.397	0.399
S	0.0412	0.0514	0.0466	0.0382	0.0463	0.0416	0.0380	0.0406	0.0349	0.0416
σ	0.0139	0.0164	0.0141	0.0111	0.0130	0.0112	0.00986	0.0102	0.00850	0.00969

TABLE 2.4.2

n	1	2	3	4	5
$\gamma_i = 3 + 3\alpha_i$	5,001	5,979	3,726	5,820	4,830
α_i	0,667	0,993	0,242	0,940	0,610
$\varphi(\gamma_i)\cdot 10^4$	0,186	0,00103	36,0	0,00257	0,414
$p = 1 - 3 \times \dfrac{\sum_{i=1}^{n} \varphi(\gamma_i)}{n}$	0,99994	0,99997	0,99637	0,99728	0,99780
S	—	0,00004	0,00622	0,00539	0,00482
σ	—	—	—	0,00468	0,00305

n	6	7	8	9	10
$\gamma_i = 3 + 3\alpha_i$	3,393	4,056	4,938	4,938	5,040
α_i	0,131	0,352	0,646	0,646	0,680
$\varphi(\gamma_i)\cdot 10^4$	107	10,9	0,251	0,251	0,154
$p = 1 - 3 \times \dfrac{\sum_{i=1}^{n} \varphi(\gamma_i)}{n}$	0,99280	0,99340	0,99418	0,99484	0,99536
S	0,0129	0,0119	0,0112	0,0107	0,0102
σ	0,00680	0,00551	0,00468	0,00412	0,00366

n	11	12	13	14	15
$\gamma_i = 3 + 3\alpha_i$	4,194	4,017	5,418	5,097	5,952
α_i	0,398	0,339	0,806	0,699	0,984
$\varphi(\gamma_i)\cdot 10^4$	6,33	12,6	0,0229	0,116	0,00121
$p = 1 - 3 \times \dfrac{\sum_{i=1}^{n} \varphi(\gamma_i)}{n}$	0,99562	0,99565	0,99598	0,99628	0,99652
S	0,00974	0,00926	0,00894	0,00867	0,00840
σ	0,00329	0,00295	0,00272	0,00252	0,00234

n	16	17	18	19	20
$\gamma_i = 3 + 3\alpha_i$	3,981	3,387	3,438	5,007	4,290
α_i	0,327	0,129	0,146	0,669	0,430
$\varphi(\gamma_i)\cdot 10^4$	14,4	109	93,2	0,180	4,33
$p = 1 - 3 \times \dfrac{\sum_{i=1}^{n} \varphi(\gamma_i)}{n}$	0,99646	0,99475	0,99349	0,99382	0,99406
S	0,00812	0,0106	0,0116	0,0114	0,0111
σ	0,00218	0,00275	0,00291	0,00278	0,00262

Then

$$p=1-\int_{3}^{6} Re^{-\frac{R^2}{2}}dR. \qquad (15)$$

And the most significant part (1) is calculated analytically here, while the less substantial part is calculated by means of statistical tests, with this being done even at the cost of introducing additional assumptions (the upper integration limit is assumed to be equal to 6 instead of ∞).

The results of the calculations carried out in this manner are presented in Table 2.4.2. The random numbers are the same as in Table 2.1.2.

Since here the integration interval is from 3 to 6, the random numbers α_i obtained from the interval 0-1 must be transformed into γ_i distributed in the interval 3-6:

$$\gamma_i=3+3\alpha_i. \qquad (16)$$

The calculation is then carried out in the conventional sequence, the only difference being that in calculating the integral the mathematical expectation of the integrand is multiplied by the integration interval (in the given case, by 3).

We see from the data of the table that this method was very effective. The error amounts only to 0.52%, the root mean square deviation amounts only to 0.25%, i.e., considerably less than in previous cases. When calculating by the conventional method, we find that the root mean square error amounts to 13%.

D. Use of Significant Selections

With this method we will begin discussion of special selection forms. It involves derivation of selections from that prompted by the problem itself, and multiplication of the final result by a standardizing factor — a correction factor — which offsets utilization of an incorrect distribution.

The essence of this method reduces to the carrying out of a large number of samples in the regions of greatest interest, i.e., in those regions producing the most significant results. For example, if we are interested in evaluating the effect of an opponent's missiles on some structure, we must examine cases in which the missiles strike close to this structure; however, cases of great deviation need not be considered, although in this case we do neglect small cases of destruction.

If we are interested in evaluating the work of a complex queueing system [mass service engineering], this is best accomplished with the most intensive flow of requisitions. Evaluation of a system of antitank defense is best carried out with a mass armored attack, since in this case all its weak aspects are most clearly revealed. Again, in this case no consideration (or a limited degree of consideration) is given to cases of attack by

weak forces, but these prove not to be decisive.

Let us examine this method more specifically on an example of integral calculation. Given the requirement to calculate

$$\mathcal{J}=\int_a^b f(x)\,dx. \tag{17}$$

Let ξ be a random quantity whose probability density corresponds to two conditions: $p(x) > 0$; when $a < x < b$

$$\int_a^b p(x)\,dx=1. \tag{18}$$

We can write the obvious equality

$$\mathcal{J}=\int_a^b \frac{f(x)}{p(x)}\,p(x)\,dx. \tag{19}$$

Let us consider what the mathematical expectation $f(\xi)/p(\xi)$ will equal:

$$M\left[\frac{f(\xi)}{p(\xi)}\right]=\int_a^b \frac{f(\xi)}{p(\xi)}\,p(\xi)\,d\xi=\mathcal{J}. \tag{20}$$

Consequently, as an evaluation of the sought integral we can assume

$$\mathcal{J}=\frac{1}{n}\sum_{i=1}^{n}\frac{f(\xi_i)}{p(\xi_i)}. \tag{21}$$

The dispersion of the quantity $f(\xi)/p(\xi)$ will be equal to

$$D=\int_a^b \frac{f^2(\xi)}{p^2(\xi)}\,p(\xi)\,d\xi-\mathcal{J}^2=\int_a^b \frac{f^2(\xi)}{p(\xi)}-\mathcal{J}^2. \tag{22}$$

D has a minimum when

$$p_0(x)=\frac{|f(x)|}{\int_a^b |f(x)|\,dx}. \tag{23}$$

The substitution of (23) into (22) then yields

$$D_0=\left[\int_a^b |f(x)|\,dx\right]^2-\mathcal{J}^2. \tag{24}$$

This means that if the integrand does not change sign, the dispersion is equal to zero. Utilization of Formula (23) directly prior to the start of the calculation is impossible, since for calculation with this formula it is necessary to know the sought

integral. However, it is extremely important, since, first of all, it provides an indication of the fact that the distribution of the random quantity ξ must be selected in a manner such that the largest number of points falls within those regions in which the values of the integrand are maximum and, secondly, it may be used in calculations in which the integral is determined according to a limited selection which also provides information as to the form of the function $f(x)$.

As an example

$$\mathcal{J}=\int_0^1 Re^{-\frac{R^2}{2}}dR,$$

assuming

$$P(R)=kR. \tag{25}$$

The calculations are carried out for the same integral in order to compare the effectiveness of various means of reducing dispersion. Here K is the coefficient correcting the "incorrect distribution." It is determined from the condition

$$\int_0^1 P(R)\,dR=1,\ \mathrm{K}\int_0^1 RdR=1,\quad \mathrm{K}=2. \tag{26}$$

Thus calculation of the integral is carried out according to the formula

$$\mathcal{J}=\frac{1}{2n}\sum_{i=1}^{n}e^{-\frac{\xi_i^2}{2}}, \tag{27}$$

where the numbers ξ_i are distributed with the density of Probability (25).

These can be calculated by means of the formula

$$\xi_i=\sqrt{\lambda_i}, \tag{28}$$

where λ_i are numbers distributed according to the law of equal probability in the interval 0-1. Derivation of this formula is shown in §2.2.

Calculation with Formula (28) when n = 20 yielded the following results: $\mathcal{J}=0{,}3852$, $S=0{,}0529$. Thus, the error in the determination of the integral amounted to 1.98%; the root mean square error according to Formula (2.3.8), amounted to 3%, i.e., a significant improvement in accuracy takes place here as well.

E. Selection by Groups

Assume that we are required to calculate the probability of target damage by bombing from bombers in any weather. One of the logical methods of solving this problem is solution of the problem

by groups: each group for a different kind of weather, but identical weather within the group, with subsequent averaging of the derived result with consideration of the probability of a given kind of weather. In final analysis, such a problem reduces to seeking an integral. However, the integral can also be calculated with respect to *m* groups

$$\int_a^b f(x)\,dx = \sum_{k=1}^{m} \int_{a_k}^{b_k} f(x)\,dx. \qquad (29)$$

If each of the integrals in the sum is calculated by the simplest method of statistical tests, then

$$\mathcal{J} = \sum_{k=1}^{m} \frac{l_k}{n_k} \sum_{i=1}^{n_k} f(\xi_{i,k}), \qquad (30)$$

where

$$l_k = b_k - a_k,\quad D = \sum_{k=1}^{m} \frac{l_k^2 D_k}{n_k}, \qquad (31)$$

$$D_k = D\,[f(\xi_k)] = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f^2(x)\,dx - \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f(x)\,dx. \qquad (32)$$

If we establish the total number N of tests, it is not difficult to demonstrate that for minimum D it is necessary to select n_k proportional to $l_k\sqrt{D_k}$. In this case

$$D = \frac{\left(\sum_{k=1}^{m} l_k \sqrt{D_k}\right)^2}{N}. \qquad (33)$$

However, the values of D_k at the beginning of the calculation are unknown. In this case we generally assume n_k proportional to l_k. In this case

$$D = \frac{b-a}{N} \sum_{k=1}^{m} l_k D_k. \qquad (34)$$

However, a successive method may be employed to determine, on the basis of a limited number of samplings, the dispersions in accordance with which the remaining samplings are divided.

Let us consider the problem of the effect of the number of groups on the accuracy of calculating the integral when l_k and n_k, as well as D_k, are equal to each other. In this case

$$D = \frac{(b-a)^2}{N} D_k, \qquad (35)$$

i.e., the dispersion of the integral is a function of the number of groups only in terms of the quantity D_k. For a qualitative evaluation of the effect of the number of groups on D_k let us consider the case

$$f(x)=Kx^n. \tag{36}$$

Then, in each group

$$M[f(x)]=\frac{1}{l_k}\int_0^{l_k} f(x)\,dx=\frac{1}{l_k}\int_0^{l_k} Kx^n dx=\frac{Kl_k^n}{n+1}, \tag{37}$$

$$D_k=\frac{1}{l_k}\int_0^{l_k}\{f(x)-M[f(x)]\}^2\,dx=$$

$$=\frac{1}{l_k}\int_0^{l_k}\left(Kx^n-\frac{Kl_k^n}{n+1}\right)^2 dx=$$

$$=K^2 l_k^{2n}\frac{n^2}{(2n+1)(n+1)^2}=\Delta f_k^2\,\varphi(n), \tag{38}$$

where Δf_k is the increment of the function in the group

$$\varphi(n)=\frac{n^2}{(2n+1)(n+1)^2} \tag{39}$$

and is dependent on the form of the function. Its form is shown in Fig. 2.4.3.

Thus, the smaller the increment of the function in the group, i.e., the smaller the group, the smaller the dispersion. At the limit we come to the groups consisting of a single point. This selection is known as *systematic*.

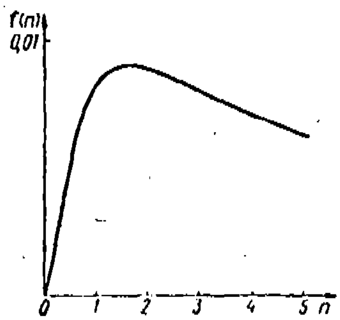

Fig. 2.4.3

TABLE 2.4.3

Group number	$\mathcal{J}$	Error in %	Root mean square error in determination of J in % of J
1	0.4240	7.9	9.4
2	0.4140	5.3	5.2
5	0.4030	2.6	2.2
20	0.3931	0.03	0.5*

*Tentatively, proceeding from Formula (38).

As an example we have calculated the integral $\int_0^1 Re^{-\frac{R^2}{2}}\,dR$ for various numbers of groups and a total number of realizations equal to 20. The results of these calculations are shown in Table 2.4.3.

We see from the table that this method may be very effective. The root mean square errors are approximately inversely proportional to the number of groups, which follows from Formula (38).

F. The Utilization of Dependent Quantities

Let it be required that a comparison be carried out of the firing accuracy of two rockets (S_1 and S_2), guided only during the powered phase, with the automatic range controls designed variously.

The range deviation for each of the rockets can be determined in the following manner:

$$x_1 = x_{a_1} + x_{n_1}, \tag{40}$$

$$x_2 = x_{a_2} + x_{n_2}, \tag{41}$$

where x_{a_1} and x_{a_2} are the deviations of the powered phase, i.e., errors due to the automatic range unit;

x_{n_1} and x_{n_2} are deviations of the unpowered phase.

Let $\sigma_{x_{a_1}} = 1$, $\sigma_{x_{a_2}} = 0{,}5\sigma_{x_{n_i}} = \sigma_{x_{n_2}} = 1$. To calculate S_1 and S_2 by the method of statistical tests, we have to determine the random realizations x_{a_1}, x_{a_2}, x_{n_1} and x_{n_2} and then x_1 and x_2 by means of Formulas (4) and (41).

If we carry out the calculations of x_{a_2}, x_{a_1}, x_{n_1} and x_{n_2}, and consequently, of x_1 and x_2, proceeding from the independent quantities, we derive the following picture (Table 2.4.4).

TABLE 2.4.4

n	1	2	3	4	5	6	7	8	9	True value of S
x_1	0.13	0.07	1.57	−0.67	−0.74	1.25	1.51	0.68	−0.92	—
x_2	−1.82	0.40	2.42	−0.70	2.30	−0.37	1.56	−0.96	1.09	—
S_1	—	0.04	0.85	0.93	0.93	0.96	0.99	0.92	0.98	1.41
S_2	—	1.00	2.16	1.80	1.90	1.73	1.65	1.61	1.53	1.11

We see from the table that the S obtained in the second case are considerably larger than in the first case, and it is thus impossible in 9 tests to select the best rocket (an incorrect selection may be made). It is possible to make this problem easier by using dependent random quantities (in the given case, identical quantities are best of all) to calculate x_{n_1} and x_{n_2}. In this case we will have the following situation (Table 2.4.5).

TABLE 2.4.5

n	1	2	3	4	5	6	7	8	9	True value of
x_1	0.13	0.07	1.57	−0.67	0.74	1.25	1.51	0.68	−0.92	—
x_2	0.61	−0.84	0.76	−0.73	0.53	1.35	2.05	0.51	−0.84	—
S_1	—	0.04	0.35	0.93	0.93	0.96	0.99	0.92	0.98	1.41
S_2	—	1.02	0.89	0.85	0.78	0.88	1.05	0.98	1.01	1.11

Selection of the best rocket in the given case from 9 tests is also impossible, but no basic error is assumed here. On the basis of the calculational result, the rockets are equal. Moreover, fewer random quantities were needed in this case.

This example has no practical significance, since it was clear without calculation which of the rockets was better. Its only purpose was to show the essence of the method.

However, the basic idea of this example is of practical significance. Let the points of incidence for two types of rockets be determined experimentally and given that we know nothing in advance as to the scattering of each rocket type. If we test the rockets under various weather conditions (this corresponds to the selection of independent random numbers to determine x_{n_1} and x_{n_2}), we obtain the picture presented in Table 2.4.4. If the tests of each pair of rockets for both types are carried out under identical weather conditions (which corresponds to the case of identical random numbers for the calculation of x_{n_1} and x_{n_2}), we obtain the situation shown in Table 2.4.5, and can solve the problem of selecting the best rockets with a limited number of tests.

An analogous situation may be encountered in testing the mathematical model, when the result is obtained by calculation with a rather complex system of equations which cannot be analyzed, and where it becomes necessary to compare two relatively large fluctuating quantities to calculate a small quantity.

In this case the comparison is best carried out, all other conditions being equal, by using dependent (or even identical) quantities. In this case, of course, a sufficiently large range of changes in conditions must be encompassed, because the variant suitable under certain conditions may prove to be less suitable under other conditions. One of the expressions of this method is the requirement to remove everything extraneous from the model.

Earlier we cited a case involving the utilization of dependent quantities for comparison of the root mean square deviations. This case may be used with equal success to compare mathematical expectations, probabilities and integral calculations.

The utilization of dependent quantities may prove to be extremely effective in comparing results of exact and applied theory.

In conclusion, let us present an example of the application of dependent quantities for the calculation of the integral. Let there be a requirement for the calculation

$$\mathcal{J}=\int_0^1 Re^{-\frac{R^2}{2}}dR=\int_0^1 \varphi(R)\,dR. \tag{42}$$

Since the integrand in the subject interval increases monotonically, with a large number of points falling on the initial part of the interval, we obtain a reduced value for the integral; in the opposite case, the integral value is exaggerated. To avoid this situation, we will use the dependent quantities. Having obtained the random number α_i, we will determine the number $1 - \alpha_i$. This also achieves uniform distribution of the numbers over the integration interval. We calculate the integral with the formula

$$\mathcal{J}=\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi(\alpha_i)+\varphi(1-\alpha_i)}{2}. \tag{43}$$

This method is occasionally referred to as the *symmetrization* of the integrand.

Calculation with this method for 20 realizations leads to an error of 6.9% in the integral, and the root mean square error amounts to 3.2%. This is a significant improvement in the accuracy of the method. Combination of the symmetric selection with separation into 5 groups permitted reduction of the error in the integral to 0.3% for a root mean square error of 0.35%.

With respect to more complex models, this method may involve, for example, the calculation of the realizations for the subject process in the case of symmetric deviations from the mean statistical temperature value.

G. Additional Remarks

Above we presented certain methods permitting significant reduction in dispersion. In certain cases this reduction in dispersion may amount to 10^6 times. A characteristic circumstance in this case is the fact that with utilization of these methods the laws of "conservation of value" are not observed, if we can express ourselves in this way. In other words, it is possible to achieve a significant reduction in dispersion either entirely without increasing the scope of the calculation, or with a minimum increase in this scope, involving absolutely no comparison with the reduction in dispersion achieved. As a rule, this is not the case in conventional numerical analysis.

Of course, the cited formulas can rarely be used directly. It is more important and advisable to employ these methods and combinations of them. The most effective may be those methods based on the utilization of specific features of the problem, as can be seen from the example of the transformation

$$\int^{3} Re^{-\frac{R^2}{2}}\,dR \cong 1 - \int_{3}^{6} Re^{-\frac{R^2}{2}}\,dR, \qquad (44)$$

in which it was possible to reduce dispersion by a factor of 2700. The cited methods by no means exhaust the possibilities of reducing dispersion.

One of the means of significantly reducing the amount of calculational work is the application of two-, three-stage selection and, finally, a method of sequential statistical analysis, with the scope of the book not permitting a detailed examination of these methods.

Another interesting way is the use of nonrandom numbers. We dwelt on the application of correlated quantities, as well as on the nonrandom division of the integration interval. Both lead to nonrandom numbers within limits. In calculating the integrals, instead of uniformly distributed random numbers, we can use the Kholton [sic] sequence [12] which for sufficiently smooth functions ensures the proportionality of the root mean square deviation of the quantity $\ln^n N/N$ instead of $1/(N)^{-2}$ for the use of random numbers (where n is the multiplicity of the integral and N is the number of tests), i.e., higher accuracy (when n = 1, N = 10 root mean square deviations are proportional to 0.230 and 0.320, respectively; when N = 100 the root mean square deviations are proportional to 0.046 and 0.100, respectively).

TABLE 2.4.6

	Calculation Method	Error from determined quantity in %	Root mean square error in %	Possible reduction in number of realizations at the given accuracy
1	$r_i = \sqrt{x_i^2 + z_i^2}\; r_i \gtrless r$	25	27.5	1
2	$\int_0^1 Re^{-\frac{R^2}{2}}\,dR = \frac{m}{n}$	14.5	27.5	1
3	$\int_0^1 Re^{-\frac{R^2}{2}}\,dR = \frac{1}{n}\sum_{i=1}^{n} \varphi(\xi_i)$	7.9	9.9	9
4	Variant No. 1 in analysis of results during the course of the realizations	3.3	11.8	5
5	$\int_0^1 Re^{-\frac{R^2}{2}}\,dR = \int_0^1 f(R)\,dR + 0.3$	1.5	4.4	60
6	Significant selection . . .	2.0	3.0	80
7	By 2 groups	5.3	5.2	30
	By 5 groups	2.6	2.2	160
	By 20 groups	0.03	—	3000
8	Symmetrization	6.9	3.2	70
9	Division into 5 groups and symmetrization	0.3	0.35	6000

In conclusion, we will present certain summary results, admittedly, without evaluation of the calculation time which for such simple examples is conditional. In all cases we consider the probability of target damage for $r = \sigma$ and 20 tests.

The above table shows that the various methods of reducing dispersion, and the combinations of these methods even more so, may yield a significant effect. Of course, under various conditions different methods may prove to be more effective and the table cannot therefore be regarded as a characteristic of the methods that is valid in all cases.

§2.5. APPLICATION OF THE METHOD OF STATISTICAL TESTS TO DETERMINE FIRING ACCURACY

A. Features of the Problem

As indicated in Chapter 1, firing accuracy is generally determined experimentally. However, the expense of tests to determine firing accuracy, the need to carry out a large number of such tests for reliable determination of firing accuracy under various conditions, etc., forces us to seek theoretical methods of determining firing accuracy. One such objective method is statistical modeling of missile flight with consideration of perturbations affecting same [35]. It is this method that is discussed below as an example of the application of the method of statistical tests to determine armament characteristics.

In this case the problem is divided into 4 stages: determination of perturbations; determination of the system of equations describing the process of missile motion; statistical tests, each of which determines the deviation of the missile from the target; statistical processing of resulting data and evaluation of the accuracy of the results derived.

Among the perturbations affecting a missile we should include:

weather disturbances;

perturbations associated with target maneuvering;

perturbations associated with fabrication inaccuracies and inaccuracies of unit operation;

electrical interference.

The problem of weather disturbances will be considered in §8.2.

The problem of possible target maneuvers is rather indeterminate. In the general case target motion may be described by the coordinates of its center of mass, which are random functions of time. Determination of the form of these functions requires analysis of the tactics involved in the utilization of corresponding facilities and their maneuvering potentials.

Perturbations associated with inaccuracies of fabrication

and operation or deviation from rated characteristics;

aerodynamic characteristics (the axial force factor C_{τ}, the normal force factor C_n, the coefficients for the stabilizing and damping moments, the appearance of aerodynamic eccentricity α_a);

engine characteristics (thrust $\mathcal{P}$, engine operating time t_k, per-second flow rate G_{sek}, appearance of gasdynamic eccentricity α_g);

weight and dimension characteristics of the rocket (rocket weight mg, inertial moments of the rocket, distance from the center of mass to the point of control-force application, midsection area,[3] etc.);

parameters of the control system, for example, the control-system gain, the servomechanism time constant, etc.

The number and nature of these parameters depend in great measure on the design of the system.

B. A System of Equations for Solutions on a Computer

The system of equations describing the motion of a guided missile can be divided into the following groups of equations:

1. The equations of target motion

$$x_{\text{цели}}=f_1(t), \; y_{\text{цели}}=f_2(t). \tag{1}$$

In the special case in which the target is stationary, these equations degenerate into target coordinates.

2. Equations of preparation of initial conditions. This group must define those initial conditions under which launch is accomplished and the conditions for the installation of corresponding devices in the missile. In particular, for guided "air-to-ground" missiles an important initial condition is the angle of the velocity vector. An error in the initial direction of the missile requires additional acceleration to guide the missile to the target.

3. Equations of motion of the center of mass and the axes of the missile. These are ordinary equations of motion for a solid body under the action of a reaction force, for aerodynamic forces and the forces of gravity in a perturbed atmosphere.

4. Equations of the guidance system are determined entirely by the basic scheme and design of the guidance system, differing significantly for the cases of remote control, autonomous control and homing. The equations of this group must make it possible to determine commands to the controls as a function of the mutual locations of rocket and target.

5. Equations of device operation which make it possible to determine the instant of warhead detonation.

[3]/163

The method of carrying out the statistical tests to determine firing accuracy is basically no different from that employed in other cases. For each test we determine the weather condition, the characteristics of the rocket assemblies; initial conditions are prepared and by integration of the system of equations describing missile motion, we determine the missile miss distance at the instant that the detonating devices are actuated.

Thus, in statistical modeling it is necessary to test a rather large number of missiles "in flight." For each circumstance and for each set of missile parameters we have to make a large number of decisions in order to take into consideration the changes in statistical quantities from flight to flight. The accuracy of the derived results is determined in the manner demonstrated in §2.3.

We note that rocket flight can be modeled by means of electronic digital computers and by means of electronic models (analog computers). There is no need in this case to speak of manual calculation because of the complexity of the systems of differential equations describing missile motion.

Existing electronic models do not require great expenditure of machine time, and they are more easily readied for operation. Their drawback is the lower accuracy (requiring transformation of the equations into different equations) and the limitations on problem complexity with respect to nonlinear terms.

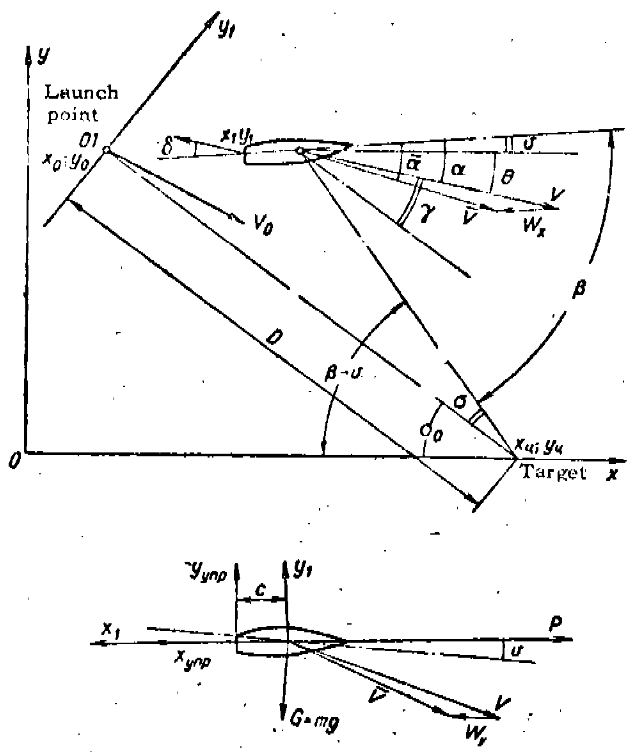

Fig. 2.5.1

Digital computers are highly accurate and exhibit extensive potentials for the solution of complex systems of equations, and these are limited primarily by expenditures of machine time and the time spent on programing.

In principle a combination of digital and analog computers is possible. As an example, let us consider the modeling of the firing of an "air-to-ground" missile. Let the missile be launched from a specific point in space, and further, let the missile be guided by means of a passive infrared homing system to the ground target. For simplicity, we will consider the plane problem, i.e., assume that the missile is moving in the vertical plane passing through the launch point and the target.

A simplified diagram is shown in Fig. 2.5.1. First of all, let us examine the system of equations for solution on a digital computer. The equations of perturbed missile motion may be written in the following form (we are considering the problem of plane motion):

$$\frac{dv}{dt}=\frac{\mathcal{P}-X_{\text{упр}}-X_1-(Y_1+Y_{\text{упр}})\alpha}{m}-g\sin\theta, \tag{2}$$

$$\frac{d\theta}{dt}=\frac{(\mathcal{P}-X_{\text{упр}}-X_1)\alpha}{mv}+\frac{Y+Y_{\text{упр}}}{mv}-\frac{g\cos\theta}{v}, \tag{3}$$

$$\mathcal{J}_z\frac{d^2\vartheta}{dt^2}+M_z^{\omega}\frac{d\vartheta}{dt}+M_z^{\alpha}\bar{\alpha}+cY_{\text{упр}}+M_z^{\alpha}\alpha_a+\mathcal{P}\alpha_{\text{г}}=0, \tag{4}$$

$$\frac{dx}{dt}=v\sin\theta, \tag{5}$$

$$\frac{dy}{dt}=v\cos\theta, \tag{6}$$

$$\alpha=\vartheta-\theta, \tag{7}$$

$$\bar{\alpha}=\alpha-\frac{W_x\sin\theta}{v-W_x\cos\theta}, \tag{8}$$

$$\bar{v}=v-W_x\cos\theta,\ \mathrm{M}=\frac{\bar{v}}{a}, \tag{9}$$

$$a=\sqrt{\mathrm{K}gR\tau}, \tag{10}$$

$$\rho=\frac{h_0 e^{-\frac{1}{2}\int_0^y \frac{g\,y}{\tau}}}{gR\tau}, \tag{11}$$

$$X_1=\frac{\rho\bar{v}^2}{2}S_{\text{м}}[C_{x_0}(\bar{\mathrm{M}})+C_n^{\alpha}(\bar{\mathrm{M}})\,\bar{\alpha}^2], \tag{12}$$

$$Y_1=\frac{\rho\vec{v}^2}{2}S_{\text{м}}C_n^{\alpha}(\bar{\mathrm{M}})\,\bar{\alpha}, \tag{13}$$

$$X_{\text{упр}}=f(\delta)\ \text{for any surfaces}, \tag{14}$$

$$Y_{\text{упр}}=f(\delta)\ \text{for any surfaces}, \tag{15}$$

$$\mathrm{M}_z^{\alpha}=C_m^{\alpha}(\bar{\mathrm{M}})\,\frac{\rho\vec{v}^2}{2}S_{\text{м}}l, \tag{16}$$

$$\mathrm{M}_z^{\omega}=C_{\text{д}}(\bar{\mathrm{M}})\,\frac{\rho\vec{v}^2}{2}\,\frac{S_{\text{м}}l^2}{v}, \tag{17}$$

$$\tau_1\dot{\delta}+\delta=K_1(\sigma+\alpha+\gamma+\eta)=K_1(\beta+\eta), \qquad (18)$$

$$\delta\leqslant\delta_{\text{доп}}, \qquad (19)$$

$$\beta=\vartheta+\operatorname{arc\,tg}\frac{y}{x_{ц}-x}. \qquad (20)$$

This system must be integrated for the initial condition (t = 0):

$$v=v_0, \qquad (21)$$

$$\vartheta=\vartheta_0, \qquad (22)$$

$$\theta=\theta_0, \qquad (23)$$

$$x=x_0, \qquad (24)$$

$$y=y_0, \qquad (25)$$

$$\dot{\vartheta}=\dot{\vartheta}_0. \qquad (26)$$

Depending on the conditions of the problem, these quantities may either be fixed or random. For example, if we are considering target damage probability under any conditions possible in combat, preliminary analysis of the nature of the combat and the launch conditions, we have to determine the statistical characteristics of the quantities $x_0, y_0, \theta_0, \vartheta_0, v_0$ and $\dot{\vartheta}_0$ and regard them as random quantities.

If we consider target damage probability from a given point, the random quantities will be v_0, θ_0, ϑ_0 and $\dot{\vartheta}_0$, since they are functions of the descent of the missiles, and the quantities x_0 and y_0 must be assumed to be constant.

If we assume that the subject missile is fitted out with a contact fuze, for the boundary condition we may assume

$$y=0 \qquad (27)$$

and under these conditions determine the coordinate x_p of the incidence point and the miss distance

$$\delta x=x_{ц}-x_{п}. \qquad (28)$$

If the fuze is of the noncontact variety, instead of the condition y = 0 we must write the equation for fuze operation.

The following denotations have been adopted in the system of equations cited above:

v – velocity of the center of missile mass relative to a nonmoving atmosphere;

$\bar{v}$ – velocity of center of missile mass relative to the actual atmosphere (with consideration of wind speed);

t – time, argument;

$\mathcal{P}$ – reaction engine thrust, a random function of time;

X_{upr} & Y_{upr} – projection of control force (gas or aerodynamic control surfaces, rotating engine, etc.), random functions of the angle of control rotation;

X_1 and Y_1 – axial and normal projections of the force of air resistance;
α – angle of attack;
$\overline{\alpha}$ – angle of attack with consideration of wind speed;
ϑ – angle of missile axis with the horizon;
θ – angle of velocity vector with the horizon;
m – mass of the rocket, a random function of time;
g – acceleration of the force of gravity, constant for a given latitude and a variable, if we consider a large range of various latitudes;
J_z – inertial moment of the rocket, a random function of time;
M_z^{ω} – derivative of the damping moment;
M_z^{α} – derivative of the stabilizing moment;
c – distance from center of mass to point of control-force application, a random function of time;
α_a – angle of aerodynamic eccentricity, a random quantity;
α_g – angle of gasdynamic eccentricity, a random quantity;
x, y – coordinates of the missile center of mass;
$\overline{M}$ – Mach number with consideration of wind speed;
a – speed of sound;
K – adiabatic exponent of the atmosphere;
R – atmospheric gas constant;
τ – air temperature, a random function of the coordinates;
ρ – air density;
h_0 – ground air pressure, a random quantity;
S_m – cross sectional area of the midsection, a random quantity;
$C_{x_0}(\overline{M})$ – coefficient of axial aerodynamic force, a random function of $\overline{M}$;
$C_n^{\alpha}(\overline{M})$ – coefficient of normal aerodynamic force, a random function of $\overline{M}$;
δ – angle of control unit rotation;
$C_m^{\alpha}(\overline{M})$ – coefficient of stabilizing moment, a random function of $\overline{M}$;
$C_{д}(\overline{M})$ – coefficient of damping moment, a random function of $\overline{M}$;
l – characteristic longitudinal missile dimension;
τ_1 – servomechanism constant, a random quantity;
K_1 – gain, a random quantity;
η – angular error in determination of missile-target line, a random function of time;
W_x – horizontal projection of wind, a random function of the coordinates and of time.

Thus, for each test we must have 13 realizations of random functions, which requires about 200 random numbers. This example clearly shows the importance of the simple derivation of random numbers.

Integration of the system of equations written out also consumes considerable time because of its complexity and the problem of applying this method may therefore frequently be governed by machine time.

C. Features in the Solution of Problems on Electronic Models

It is considerably simpler but less exact for the subject problem to be solved on electronic models. For this, first of all, the system of differential equations describing the motion of the missile must be simplified.

We assume the missile velocity and air density to be constant and we select new coordinate axes x_1, o_1, y_1, for which the o_1x-axis is directed from the launch point at the target. Moreover, we will assume the rocket mass to be constant and the angles α to be small, and also that $W_x = 0$.

In this case the system of equations describing the motion of the rockets is considerably simplified.

In Eq. (2) we can neglect $(Y_1 + Y_{upr})\alpha$ when, considering that $\partial v/\partial t = 0$:

$$\frac{\mathcal{P} - X_{\text{упр}} - X_1}{m} = g\sin\theta. \tag{29}$$

Substituting (29) into Eq. (3), we obtain

$$\frac{d\theta}{dt} = \frac{g\sin\theta\cdot\alpha}{v} - \frac{g\cos\theta}{v} + \frac{Y_1 + Y_{\text{упр}}}{mv}. \tag{30}$$

We see from this equation that the first term is small in comparison with the second. Consequently, it can be neglected. However, the second term is also small in comparison with the third. The quantity Y_{upr} is generally smaller than Y_1 by an order and it may therefore be neglected. Then

$$\frac{d\theta}{dt} = \frac{d\gamma}{dt} = \frac{Y_1}{mv} = \frac{\alpha}{K_s}, \tag{31}$$

where

$$K_s = \frac{2mv}{\rho v^2 C_n^\alpha(M)}. \tag{32}$$

On the basis of Fig. 2.5.1 we can write

$$m\frac{d^2y_1}{dt^2} = -gm\cos\sigma_0 + (Y_1 + Y_{\text{упр}})\cos\gamma. \tag{33}$$

Hence, neglecting Y_{upr}

$$\frac{d^2y_1}{dt^2} = \frac{\rho v^2 S_M C_n^\alpha(M)\cos\gamma\cdot\alpha}{2m} - g\cos\sigma_0,$$

or considering (32),

$$\frac{d^2y_1}{dt^2} = v\frac{d\gamma}{dt}\cos\gamma - g\cos\sigma_0. \tag{34}$$

We now transform Eq. (4), considering (7) and assuming

$$\bar{Y}_{\text{упр}} = K_4 \delta,$$

$$\mathcal{J}_z \frac{d^2\alpha}{dt^2} + \mathcal{J}_z \frac{d^2\theta}{dt^2} + M_z^{\omega} \frac{d\alpha}{dt} + M_z^{\omega} \frac{d\theta}{dt} + $$

$$+ M_z^{\alpha} \alpha = cK_4 \delta - M_0, \tag{35}$$

where

$$M_0 = M_z^{\alpha} \alpha_a + \mathcal{P} \alpha_r; \tag{36}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{K_3} \frac{d\alpha}{dt}. \tag{37}$$

Substituting the values from (31) and (37) into (35), we obtain

$$\mathcal{J}_z \frac{d^2\alpha}{dt^2} + \mathcal{J}_z \frac{1}{K_3} \frac{d\alpha}{dt} + M_z^{\omega} \frac{d\alpha}{dt} + \frac{M_z^{\omega}}{K_3} \alpha + M_z^{\alpha} \alpha = cK_4 \delta - M_0$$

or

$$\frac{d^2\alpha}{dt^2} + K_5 \frac{d\alpha}{dt} + K_6 \alpha = K_2 \delta - M_1, \tag{38}$$

where

$$K_5 = \frac{\frac{\mathcal{J}_z}{K_3} + M_z^{\omega}}{\mathcal{J}_z}; \tag{39}$$

$$K_6 = \frac{\frac{M_z^{\omega}}{K_3} + M_z^{\alpha}}{\mathcal{J}_z}; \tag{40}$$

$$K_2 = \frac{cK_4}{\mathcal{J}_z}; \tag{41}$$

$$M_1 = \frac{M_0}{\mathcal{J}_z}. \tag{42}$$

Thus, the following system of equations has been determined:

$$\tau_1 \dot{\delta} + \delta = K_1 (\sigma + \alpha + \gamma + \eta), \tag{18}$$

$$\delta \leqslant \delta_{\text{доп}}, \tag{19}$$

$$\frac{d^2\alpha}{dt^2} + K_5 \frac{d\alpha}{dt} + K_6 \alpha = K_2 \delta - M_1, \tag{38}$$

$$\frac{d\gamma}{dt} = \frac{\alpha}{K_3}, \tag{32}$$

$$\frac{d^2 y_1}{dt^2} = v \frac{d\gamma}{dt} \cos\gamma - g \cos\sigma_0, \tag{34}$$

to which we have to add the obvious equalities

$$\sigma = \frac{y_1}{D - x_1}, \quad x_1 = vt.$$

Integration of the system must be carried out through $x_1 = D$. The initial conditions are (21)-(26).

The block diagram for the mathematical modeling of this problem is shown in Fig. 2.5.2. In this case the process of statistical modeling consists in the following:

1. The random realizations of the following random quantities are determined: $K_1, \tau, v, c, \mathcal{J}_z, K_4, m, \rho, S_M, \sigma_n^\alpha(M), M_z^\omega, M_z^\alpha, M_0$. Since a constant velocity, density and mass have been assumed, all random functions (except η) degenerated into random quantities. Scattering of the mean rocket velocity must be calculated in advance, proceeding from the scattering of the rocket and atmosphere characteristics.

2. We calculate the coefficients contained in the equations, i.e., K_2, K_3, K_5, K_6, M_1 from the appropriate formulas for each realization.

3. We carry out the modeling as a result of which we determine the miss distance δy_1 which, according to the formula $\delta y = \delta y_1/\sin\sigma_0$ is converted into the miss distance at the ground surface.

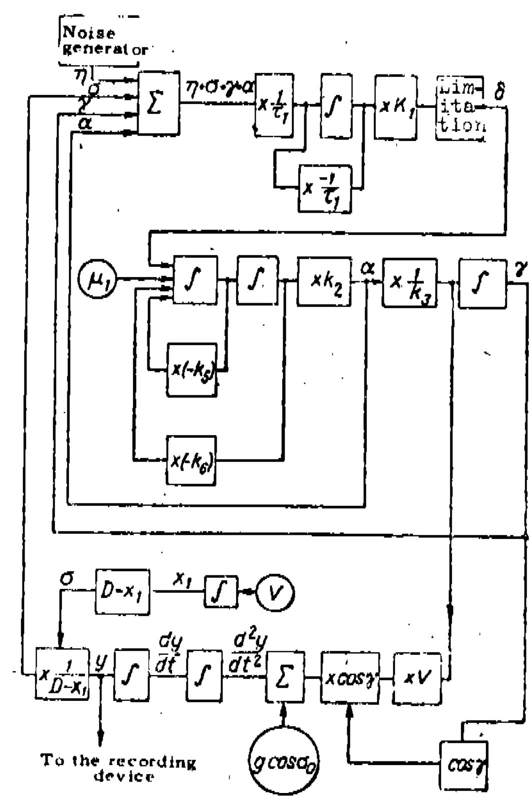

Fig. 2.5.2

4. We carry out the statistical processing of the results and evaluate their accuracy.

In speaking of an evaluation of accuracy, we should bear in mind that in this case there is a methodological error which may

attain significant magnitudes. It consists of two components. The first is a result of the limited accuracy of the electronic models. In practical terms, it can be determined by multiple repetition of the modeling under completely identical conditions, with subsequent statistical processing of the results.

The second component is a result of the simplification of the system of equations, i.e., of the assumptions adopted. It can be evaluated by comparing the results of modeling on an electronic model with the results from the modeling on a digital computer.

A third method of modeling is possible and consists of a combination of mathematical and physical modeling. Its essence calls for the entire rocket to be included in the modeling contour. The angle of rotation for its control surfaces is picked up by means of special devices and included in the mathematical model at the output of which we obtain the angle σ which enters a special simulator which moves a mock-up of the target in front of the head. This modeling makes possible exact consideration of all features involved in the operation of the internal rocket contour, but to obtain objective data many rockets have to be tested in order to be able to take into consideration the effect of the scattering of their characteristics.

Manuscript Page No.	Footnotes
102	[1]We proceed in this manner because $b - a$ in the given case is equal to 1 and y is taken as equal to 1. Otherwise, we would have to calculate $\alpha_i = (b - a)\delta_{1,i}$ and $\beta_i = y\delta_{2,i}$.
102	[2]We proceed in this way because $b - a = 1$ and $y = 1$.
155	[3]Мидель is taken from the Dutch "middel" – the middle and widest portion of a vessel.

Manuscript Page No.	Transliterated Symbols
124	си = si = statisticheskiye ispitaniya = statistical tests
124	м = m = metodicheskiy = method(ical)
128	в = v = verkhnyy = upper
128	н = n = nizhnyy = lower

129 ср = sr = sredniy = average [mean]

155 сек = sek = sekundnyy = per second

155 г = g = gazodinamicheskiy = gasdynamic

155 цели = tseli = tseli = target

157 упр = upr = upravlyayushchiy = controlling

157 доп = dop = dopolnitel'nyy = additional

158 п = p = padeniya = incidence

158 ц = ts = tsel' = target

159 м = m = midel' = midsection

Chapter 3

EVALUATION OF FIRING EFFECTIVENESS FOR A SINGLE WEAPON FORM

§3.0. INTRODUCTION

This chapter deals with various cases of evaluating the firing effectiveness of a single weapon at one or more targets. The term weapon is understood to refer to firearms, artillery and rocket installations of various designations.

The first three sections are devoted to an evaluation of effectiveness for a single shot and for various forms of the damage probability. In §3.4 we have an evaluation of firing effectiveness in the case of an arbitrary number of independent shots at a target and for an exponential damage probability. Here we present the necessary expressions to calculate the damage probability of a target. In §3.5 we consider an evaluation of firing effectiveness for dependent shots. We deal with the relationship between shots in the presence of individual errors and errors in the preparation of initial firing data (in a scheme of two firing error groups).

The following section (3.6) is also devoted to an evaluation of firing effectiveness for the case of dependent shots. However, here the relationship between the shots is considered not from the standpoint of the presence of firing errors, but from the standpoint of weapon (launching installation) unreliability. The expressions presented in this section permit consideration of weapon reliability characteristics in the evaluation of weapon effectiveness.

The last section (3.7) is devoted to an evaluation of the effectiveness of a single weapon in the combat situation in which it becomes necessary to take into consideration not only individual random factors, but also the simultaneous action of all of these factors (accuracy, reliability, scanning characteristics, survivability, etc.).

The effectiveness of a single weapon in a combat situation is evaluated by means of the total target damage probability ($\tilde{R}_1$) for a single shot or for n shots ($\tilde{R}_n$).

Unlike the total target damage probability in §§3.1-3.6 we consider the conditional target damage probability (R_1) for a single shot and for n shots (R_n) (for the condition that the target has been detected and that the system functions reliably.

§3.1. EVALUATION OF EFFECTIVENESS IN IMPACT FIRING

A. Analytical Methods

Impact projectile firing is used extensively to destroy ground targets, as well as aerial targets. The use of a given projectile for firing operations is governed by target type and projectile power.

In this section we will deal with an evaluation of firing effectiveness in the simplest case – a single shot of an impact projectile at a single target.

The probability of target damage with a single shot is a complex event and analytically may be expressed as the product of the probabilities of two random events: hitting the target and damaging the target when hit

$$R_1 = P_1 G, \quad (1)$$

where R_1 is the conditional probability of target damage under the condition that the target has been detected and that the system functions reliably;
P_1 is the probability of hitting the target with a single shot;
G is the probability of damaging the target under the condition of the projectile hitting the target.

The damage probability G was dealt with in Chapter 1.

We dwell in greater detail on calculating the probability of hitting the target. The probability P_1 of hitting the target is a function of target dimension and shape, of the location of the mean flight path, the magnitudes of scattering and firing direction [10, 24].

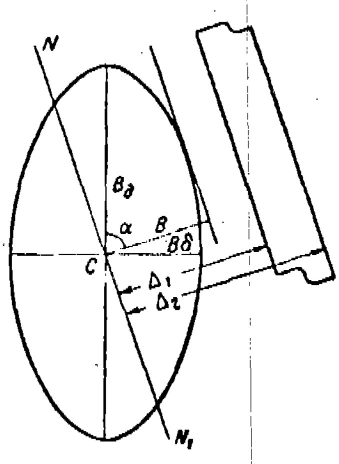

Fig. 3.1.1

Target dimensions may exceed the dimensions of the scattering ellipse with semiaxes along $3\sigma_x$ and $3\sigma_z$, where σ_x and σ_z are the standard deviations of the projectile with respect to range and in the lateral direction. In this case the probability of hitting the target will be affected most decisively by the position of the mean flight path relative to the target center. When the mean trajectory coincides with the center of the target, the hit probability will be close to unity (100%), while with deviation of the mean trajectory by $3\sigma_x$ or $3\sigma_z$ over the dimensions of the target, this probability will be close to zero.

The probability of hitting a band of infinite length with a single shot is given by the formula

$$P_1=\frac{1}{\sigma\sqrt{2\pi}}\int_{\Delta_1}^{\Delta_2} e^{-\frac{\Delta^2}{2\sigma^2}}d\Delta, \quad (2)$$

where

$$\sigma=\sqrt{\sigma_x^2\cos^2\alpha+\sigma_z^2\sin^2\alpha}; \quad (3)$$

Δ_1 is the distance from the center of projectile scattering to the nearest boundary of the band (Fig. 3.1.1);
Δ_2 is the same for the farthest boundary of the band;
$\Delta_2 - \Delta_1$ is the width of the band.

Expression (2) may be written differently in the following manner [10]:

$$P_1=F_0(\beta_2)-F_0(\beta_1), \quad (4)$$

where $\beta_2=\frac{\Delta_2}{\sigma}$; $\beta_1=\frac{\Delta_1}{\sigma}$; and $F_0(\beta)$ is a tabulated function (see Table 3 of the appendix).

When firing at a target in the form of a rectangle we determine the hit probability in two mutually perpendicular bands [see (4)], and then these probabilities are multiplied

$$P_1=P_x\cdot P_z. \quad (5)$$

In actual practice, the calculation of the hit probability in a rectangle or in a band finds extensive application, for example: in calculating P when firing at dugouts, trenches, barbed wire barriers, etc. If the damage probability G or the mean number ω of required hits are known for each of these targets, the probability of target damage with a single shot is determined from Formulas (1) or (6), where instead of G we have $1/\omega$:

$$R_1=\frac{P_1}{\omega}. \quad (6)$$

There is no exact analytical expression to calculate the probability of hitting a target of complex configuration. Various approximation methods are therefore employed. Without dwelling on a detailed discussion, we will enumerate these.

To calculate the hit probability with a single shot for a target of arbitrary shape we can use a graphical method in conjunction with a probability grid or use an approximate method of comparing the areas of the target and of a rectangle whose sides are parallel to the directions σ_x and σ_z. Occasionally, we use the so-called *coefficient of target configuration*, representing the ratio of target area to the area of the described rectangle. The probability of hitting the target is then defined as the product of the probability of hitting the rectangle by that coefficient.

At the present time, for the solution of this type of problem we can employ the method of statistical tests (see Chapter 2).

B. The Method of Statistical Tests

Let us examine the sequence for the calculation of the damage probability for an airplane with a single shot by the method of statistical tests, if the mean required number of hits is $\omega = 2$, and if the firing errors are distributed normally ($m_x = m_z = 0$, $\sigma_x = 25$ m and $\sigma_z = 17$ m). The area for the projection of the target onto the xz plane perpendicular to the relative trajectory is shown in Fig. 3.1.2.

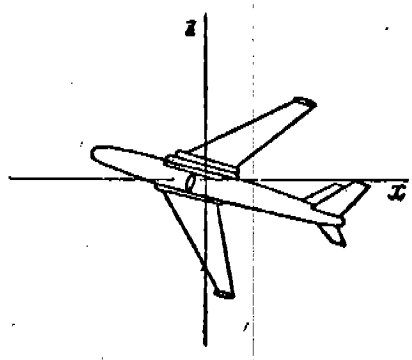

Fig. 3.1.2

The calculation sequence is the following:

1. Plot the area of the target projection onto a graph, causing the coordinate origin to coincide with the aiming point.

2. Determine the coordinates of the projectile hit point x_n, z_n:

$$\left.\begin{aligned} x_n &= \sigma_x \delta_{1,n}, \\ z_n &= \sigma_z \delta_{2,n}, \end{aligned}\right\} \qquad (7)$$

where $\delta_{1,n}$, $\delta_{2,n}$ are random numbers distributed normally over the standard deviations equal to unity and a mathematical expectation equal to zero (see Table 1 of the appendix);
n is the realization number.

3. Plot the coordinates of the hit point on the graph on which the target projection has been plotted and evaluate the hit.

If the hit point with the coordinates x_n, z_n falls within the projection area, a hit has occurred; if it goes beyond the area of the projection, there is no hit.

4. Calculate the conditional probability of target damage (or hit, in which case, without ω) from the formula

$$R_1 = \frac{m}{n_1 \omega}, \qquad (8)$$

where m is the number of projectile hits in the area of the target projection;
n_1 is the number of realizations;
ω is the mean required number of hits.

This method of calculating the hit probability for a target of complex configuration may be employed in manual calculation as well. In this case, the random numbers $\delta_{i,n}$ are taken from the table of random numbers, and the results of the calculation are summarized in a table. Table 3.1.1 shows the calculational re-

sults for a number of values of n_1.

TABLE 3.1.1

n_1	10	20	30	40	50	70	90	100
m	1	1	1	1	2	4	6	6
p	0.10	0.050	0.033	0.025	0.04	0.057	0.066	0.06

In the table m is the number of hits for n_1 shots; p is the hit frequency which with $n_1 \geq 70$ is close to the hit probability.

We see from Table 3.1.1 that the hit frequency with respect to an aircraft is $p = 0.06$ (for 100 realizations). The minimum value $p_{\text{min}} = 0.025$ was obtained for $n = 40$, while the maximum value $p_{\text{max}} = 0.1$ was obtained for $n = 10$. It is obvious that $p = 0.06$ will be the more exact value, since this value was obtained from a large number of realizations and corresponds to the middle of the range with variations in p from the minimum to the maximum value. If we assume conditionally that $\omega = 2$, from Formula (6) we find the probability of target damage with a single shot

$$R_1 = \frac{0,06}{2} = 0,03.$$

§3.2. EVALUATION OF EFFECTIVENESS IN THE CASE OF LONGRANGE FIRING

A. Exact Methods

Unlike impact firing, longrange firing is characterized by the fact that the target need not be hit directly to achieve target damage, and that in addition to the scattering of the hit points in the horizontal plane the explosions are scattered in the vertical plane as well. Thus, in the case of longrange firing we have three-dimensional scattering of the explosions in space. This scattering is subject, as a rule, to normal distribution. If all of the explosions in a plane have been distributed within an ellipse, in space the scattering of the explosion points will be ellipsoidal. The explosion ellipsoid center is known as the *center of explosion scattering*.

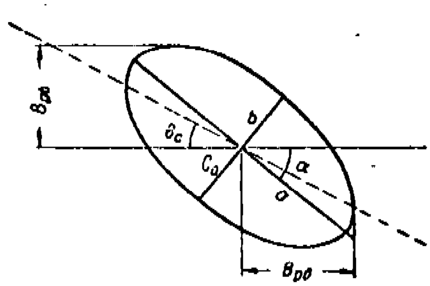

Fig. 3.2.1

In actual practice we frequently consider only two-dimensional scattering of the explosion points in the firing direction and vertically. In this case, all explosions occurring within the ellipsoid are projected onto the target plane. The distribution pattern for the explosion points in the target plane that is produced in this case is conditionally referred to as *the region of possible explosion* [10]. It may be assumed that in firing with longrange projectiles we will have two-dimensional scattering of the explosions,

since in this case the scattering in the lateral direction is considerably smaller than the longitudinal and vertical scattering, and in a number of problems this scattering may be neglected (when the width dimensions of the target exceed the scattering in the lateral direction or when the damage zone for the explosion of a single projectile exceeds the lateral scattering).

The principal sources of errors resulting in the scattering of explosions in the vertical plane are errors in determination of the angle of departure $B_{r\theta}$, of the initial velocity B_{rv} and of the fuze actuation time B_{rt}. These errors yield an elliptical error whose center is at the point C_0 (Fig. 3.2.1). In Fig. 3.2.1 we have denoted:

C_0 is the center of explosion scattering;
B_{rd} is the mean deviation of the explosions with respect to range;
B_{rv} is the mean deviation of the explosions with respect to altitude;
a is the semimajor axis of the ellipse;
b is the semiminor axis of the ellipse;
α is the angle defining the direction of the semimajor axis.

The elliptical error can be characterized by the semimajor and semiminor axes. However, in actual practice, it is more convenient to use the probable deviations caused by the unit elliptical errors B_{rd} and B_{rv} which are given in firing tables. Having determined B_{rd} and B_{rv} by means of the firing tables, we can construct a unit explosion scattering ellipse in the vertical plane. To construct the region of possible explosions we must increase the unit ellipse by a factor of 4.

After we have determined the region of possible explosions, we can take into consideration other factors affecting the probability of target damage. These factors include:

a) errors in determining initial data for firing;

b) the random nature of target damage under the condition of projectile explosion in the region of possible explosions with a deviation of r from the target.

Both of these factors are random. Each of these is therefore taken into consideration in calculating the damage probability with the corresponding probability.

The probability of target damage in firing longrange projectiles with consideration of these probabilities and the above assumptions to the effect that the three-dimensional scattering reduces to two-dimensional is determined from the formula

$$R_1 = \iint_{-\infty}^{+\infty} G(x, y) P(x, y)\, dx\, dy, \qquad (1)$$

where R_1 is the conditional probability of target damage under the condition that the target has been detected and

given the assumption that all elements of the system are functioning reliably;

$P(x,y)dx\,dy$ is the probability of hitting in the region $dxdy$;

$G(xy)$ is the probability of target damage under the condition that the projectile has exploded at a point having the coordinates x, y.

This formula is also valid for only a single shot.

If we cause the coordinate origin to coincide with the scattering center and if we direct the coordinate axes in the direction of the principal semiaxes, from the given standard deviations σ_x and σ_y of the explosion ellipse we can find the probability of the projectile hitting the region $dxdy$ (with respect to the center of explosion scattering) from the expression

$$P(x,\ y)=\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\left(\frac{x}{2\sigma_x^2}+\frac{y}{2\sigma_y^2}\right)\right]dxdy, \tag{2}$$

where x and y are the distances of the explosion from the scattering center along the directions x and y, respectively.

Let the probability of target damage for the condition of an explosion at some specific point (x,y) be equal to $G(x,y)$; the conditional probability of target damage for a projectile hit in the region $dxdy$ is then determined from the formula

$$R_1(x,\ y)=G(x,\ y)\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dxdy. \tag{3}$$

Having integrated the last expression for all points of the explosion ellipse, we can find the probability of target damage for a single shot with consideration of the explosion scattering (for the condition that the target has been detected and with reliable operation of the system)

$$R_1=\iint\limits_{-\infty}^{+\infty}\frac{G(x,\ y)}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dxdy. \tag{4}$$

We should take note of the fact that the analytical calculation of R_1 is cumbersome and it is therefore frequently carried out by means of a computer or resort is made to the utilization of various approximate calculation methods.

In firing at area targets, the mathematical expectation of the number (the relative number, percentage) of damaged targets (or target elements) is taken as the effectiveness criterion.

If the mathematical expectation of the number of damaged targets on explosion of a longrange projectile at some specific point $(x,\ y)$ is equal to $M(x,y)$, with consideration of the probability of explosion in the region $(x+dx)(y+dy)$ the mathematical expectation will be

$$M(x, y)\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dxdy. \quad (5)$$

Having integrated Eq. (5) over the entire region of possible explosions, we can find the M_1 for a single shot under the conditions of target detection and reliable system operation. The equation for the mathematical expectation of the number of damaged targets in this case will have the form

$$M_1=\iint\limits_{-\infty}^{+\infty}\frac{M(x, y)}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dxdy, \quad (6)$$

Equation (6) is solved by means of a computer, both through direct integration and by the method of statistical tests.

Graphically it is possible to find an approximate solution for Eq. (6).

B. The Method of Reference Zones

The graphical method and the method of reference zones are included among the approximate methods of calculating the probability of target damage, if the damage probability and error function are known. With the appearance of computers the graphical method lost its significance; however, the method of reference zones has found extensive application. The essence of this method involves the fact that the damage probability $G(r)$ is used to determine the reference damage radius

$$r_0=\int\limits_0^\infty G(r)\,dr, \quad (7)$$

and the hit probability for the reference area (volume) of a target is then determined analytically. In the case of longrange firing, if the dimensions of the target are small in comparison with the damage radius, for the target reference volume w_0 we assume a hemisphere having the radius r_0

$$w_0=\frac{2}{3}\pi r_0^3.$$

The probability of target damage is defined as the probability of hitting a semicircle, where we do not take into consideration ground explosions, or the probability of hitting a circle, if we take ground explosions into consideration.

The probability of hitting a circle is calculated in final analysis by making the center of explosion scattering coincide with the target and by assuming $\sigma_x = \sigma_y = \sigma$. The probability of hitting a random point within a circle in this case is equal to [24]

$$P(r)=1-e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}. \quad (8)$$

where r is the circle radius.

With consideration of ground explosions the probability of target damage with a single shot of a longrange projectile is determined with the formula

$$R_1 = 1 - e^{-\frac{1}{2}\left(\frac{r_0}{\sigma}\right)^2}, \tag{9}$$

where $\sigma = \sigma_x = \sigma_y$,

r_0 is the reference damage radius, determined from Formula (7).

When using atomic weapons against ground targets, an aerial explosion is used. Let us assume that we are employing the coordinate damage probability $G(r)$ (see §1.3). In this case, in evaluating the damage probability, the method of reference zones yields satisfactory accuracy. The probability of target damage is defined as the probability of causing an explosion within a circle having the reference damage radius. In this case the scattering with respect to altitude is neglected.

In conclusion of this section we will consider two cases of the analytical calculation of R_1 and we will provide an evaluation of the accuracy for the method of reference zones. If we denote:

O as the target;
x, y as the point of projectile [missile] explosion;
x_0, y_0 as the scattering center,

then Eq. (4) may be written differently as follows:

$$R_1 = \iint_{-\infty}^{+\infty} \varphi(x - x_0, y - y_0) G(x, y) \, dx \, dy. \tag{10}$$

We denote

$$\iint_{-\infty}^{+\infty} G(x, y) \, dx \, dy = S, \tag{11}$$

and

$$\frac{G(x, y)}{S} = \psi(x, y), \tag{12}$$

where S is the reference damage zone.

Having substituted (12) into (10), we obtain

$$R_1 = S \iint_{-\infty}^{+\infty} \varphi(x - x_0, y - y_0) \psi(x, y) \, dx \, dy = S f(x_0, y_0), \tag{13}$$

where $f(x_0, y_0)$ is the composition of the distributions φ and ψ.

Approximately,

$$(x_0, y_0) = \frac{1}{2\pi\sigma_\Sigma^2} \exp\left[-\frac{1}{2}\left(\frac{x_0^2}{\sigma_\Sigma^2} + \frac{y_0^2}{\sigma_\Sigma^2}\right)\right], \tag{14}$$

$$\sigma_\Sigma^2 = \sigma_\varphi^2 + \sigma_\psi^2, \tag{15}$$

$$\sigma_\psi^2 = \iint_{-\infty}^{+\infty} \psi(x, y)\, x^2\, dx\, dy. \tag{16}$$

CASE 1. Circular scattering. Stepped damage probability (see Fig. 3.3.1).

Let

$$\begin{cases} G(r) = 1 \text{ when } r \leqslant r_0, \\ G(r) = 0 \text{ when } r > r_0, \end{cases}$$
$$S = \pi r_0^2,$$

where r_0 is the reference radius of the damage zone

$$\sigma_\psi^2 = \iint_{-\infty}^{+\infty} \frac{G(r)}{S} x^2 dx\, dy$$

or

$$\sigma_\psi^2 = \iint_{-\infty}^{+\infty} \frac{G(r)}{S} y^2 dx dy,$$

since the distributions of x and y are identical $(x^2 + y^2 = r^2)$, so that

$$2\sigma_\psi^2 = \iint_{-\infty}^{+\infty} \frac{G(r)}{S} r^2\, dx dy = \int_0^{2\pi}\int_0^{r_0} \frac{G(r)}{S} r^3 dr d\varphi = \frac{r_0^2}{2},$$

$$\sigma_\psi^2 = \frac{r_0^2}{4}. \tag{17}$$

With

$$x_0 = y_0 = 0,$$

$$R_1 = 1 - e^{-\frac{r_0^2}{2\sigma_\Sigma^2}}, \tag{18}$$

$$R_1 = Sf(x_0, y_0)$$

or

$$R_1 = \pi r_0^2 \frac{1}{2\pi\sigma_\Sigma^2}, \tag{19}$$

where

$$\sigma_\Sigma^2 = \sigma^2 + \frac{r_0^2}{4}; \tag{20}$$

$$R_1 \approx \frac{r_0^2}{2\sigma^2 + \frac{r_0^2}{2}} = \frac{z^2}{2 + \frac{z^2}{2}}, \tag{21}$$

where $z = \frac{r_0}{\sigma}$.

Let us verify the accuracy of Eq. (21) by means of Eq. (19). The results of the calculation are shown in Table 3.2.1 for various values of

$$z = \frac{r_0}{\sigma_\Sigma}.$$

We see from Table 3.2.1 that the approximate method of calculating the probability of target damage is in good agreement with the method of reference zones when $z \leq 1$, i.e., when the reference radius for the damage zone does not exceed a single standard missile deviation. This condition is always satisfied when firing missiles with conventional warheads and in a number of cases when firing nuclear ammunition.

TABLE 3.2.1

z	$R_1 = 1 - e^{-\frac{z^2}{2}}$	$R_4 = \frac{z^2}{2 + \frac{z^2}{2}}$
0	0	0
0,1	0,005	0,005
0,5	0,117	0,118
0,8	0,274	0,276
1,0	0,393	0,400
1,5	0,675	0,720
2,0	0,865	1,0
3,0	0,99	1

CASE 2. Circular scattering. Damage probability *G(r)* shown in Fig. 3.2.2:

$$G(r) = 1 \text{ when } r \leqslant a,$$
$$G(r) = A - Br \text{ when } b \geqslant r \geqslant a.$$

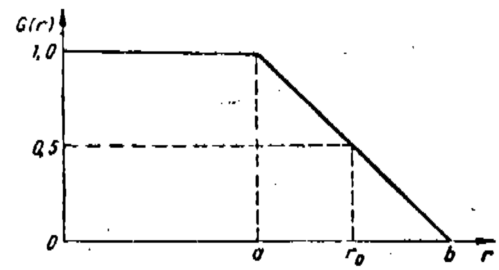

Fig. 3.2.2

We determine the reference damage zone S

$$S = \iint_{-\infty}^{+\infty} G(x, y)\, dx\, dy = 2\pi \int_0^\infty G(r)\, r\, dr =$$

$$= \pi a^2 + 2\pi \int_a^b (A - Br)\, r dr =$$

$$= \pi a^2 + A(\pi b^2 - \pi a^2) - 2B\pi\left(\frac{b^3}{3} - \frac{a^3}{3}\right).$$

We determine A and B from the condition

$$A + Bb = 1, \quad A - Bb = 0,$$

$$B = \frac{1}{b-a}, \quad A = \frac{b}{b-a},$$

$$S = \pi a^2 + \pi b(b+a) - \frac{2}{3}\pi(a^2 + ab + b^2) = \frac{\pi}{3}(a^2 + ab + b^2). \qquad (22)$$

From Eq. (16) we determine σ_ψ

$$\sigma_\psi^2 = \frac{1}{2}\iint \frac{G(r)}{S} r^3 dr d\varphi = \frac{3}{20} \cdot \frac{b^5 - a^5}{b^3 - a^3}. \qquad (23)$$

The total dispersion is determined from Eq. (15)

$$\sigma_\Sigma^2 = \sigma^2 + \sigma_\psi^2,$$

while the probability of target damage is determined from Formula (18)

$$R_1 = 1 - e^{-\frac{r_0^2}{2\sigma_\Sigma^2}}$$

or

$$R_1 = Sf(x_0, y_0).$$

When $x_0 = y_0 = 0$

$$R_1 = \frac{S}{2\pi\sigma_\Sigma^2}. \qquad (24)$$

Having substituted (23) into (24), we obtain

$$R_1 = \frac{a^2 + ab + b^2}{6\sigma^2 + 0.9\frac{b^5 - a^5}{b^3 - a^3}}. \qquad (25)$$

In conclusion, let us examine the accuracy of the method of reference zones.

For the damage probability $G(r)$ shown in Fig. 3.2.2, it seems to be possible to obtain an exact solution for R_1:

$$R_1 = \int_0^b \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} G(r)\, dr = \int_0^a \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr +$$

$$+ \int_a^b (A - Br) \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr. \qquad (26)$$

Having integrated Eq. (26), we obtain

$$R_1 = 1 - e^{-\frac{b^2}{2\sigma^2}}(A - Bb) + e^{-\frac{a^2}{2\sigma^2}}(A - aB - 1) - \\ - \sigma B\sqrt{2\pi}\left[F_0\left(\frac{b}{\sigma}\right) - F_0\left(\frac{a}{\sigma}\right)\right]. \quad (27)$$

Having used Eq. (27), we evaluate the accuracy of the method of reference zones.

Let $b = 5a$ (see Fig. 3.2.2). In this case we will have

$$r_0 = \frac{a+b}{2} = 3a, \quad B = \frac{1}{4a}, \quad A = \frac{5}{4}.$$

Having substituted r_0 into Eq. (18), we obtain R_1 for the method of reference zones

$$R_1 = 1 - e^{-\frac{4,5}{z^2}}, \quad (28)$$

where

$$z = \frac{\sigma}{a}.$$

Equation (27) under these conditions is simplified

$$R_1 = 1 - \frac{z\sqrt{2\pi}}{4}\left[F_0\left(\frac{5}{z}\right) - F_0\left(\frac{1}{z}\right)\right]. \quad (29)$$

The calculational results for the various values of z are presented in Table 3.2.2.

We see from Table 3.2.2 that the method of reference zones ensures good accuracy, since the errors of the method of reference zones do not exceed 13%. In actual practice it therefore finds extensive application.

TABLE 3.2.2

$z = \frac{\sigma}{a}$	R_1 (Reference zone method) $r_0 = 3a$)	R_1 (according to Eq. (29))
0	1.0	1.0
0.5	1.0	0.993
1.0	0.989	0.901
1.5	0.865	0.766
2.0	0.675	0.622
3.0	0.388	0.395
5.0	0.165	0.180
10.0	0.045	0.054

In evaluating the effectiveness of target damage with nuclear weapons, when $\sigma \geq 0.7\ a$, we can use Eq. (25) if we replace the damage probability $G(r)$ (§1.3) by the component damage probability (see Fig. 3.2.2). With $\sigma < 0.7\ a$, Eq. (25) does not provide for the necessary accuracy and the calculation must therefore be carried out in accordance with Eq. (27).

As a function of the stated goal of the study, the evaluation of effectiveness for a single shot may be accomplished either by employing analytical calculation methods or by using the method of statistical tests. The accuracy and time required for calculation serve as the criteria for the selection of a particular method.

§3.3. EVALUATION OF EFFECTIVENESS IN FIRING AT A POINT TARGET WITH A PROJECTILE [MISSILE] CARRYING A POWERFUL WARHEAD

In this section we will consider the case of firing at a small-scale (point) target. Here we have in mind that the dimensions of the target are small in comparison with the missile's radius of damaging effect. We are called upon to determine the probability R_1 of target damage with a single shot under the condition of the normal functioning of the firing system.

Let us examine this problem for the condition that the total firing errors are subject to circular normal distribution σ with dispersion σ^2.

Let r denote the random distance between the point of projectile explosion and the target. The firing error function in the absence of systematic errors is written in the following form:

$$\varphi(r)=\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right). \quad (1)$$

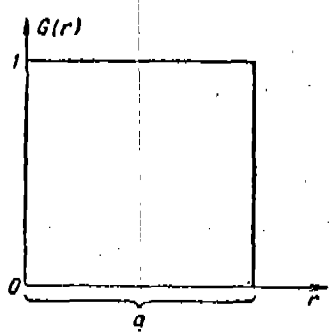

Fig. 3.3.1

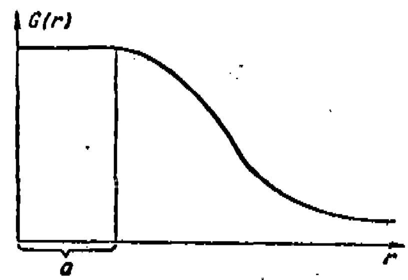

Fig. 3.3.2

Let $G(r)$ denote the damage probability, i.e., the probability of target damage under the condition that the missile has been detonated at a distance r from the target. The damage probability of the target of interest to us in this case for a single shot is then found from the equation

$$R_1=\int_0^\infty \varphi(r)G(r)\,dr. \quad (2)$$

The simplest damage probability has the form shown in Fig. 3.3.1. Here

$$\left.\begin{aligned} G(r)&=1 \text{ when } 0\leqslant r\leqslant a,\\ G(r)&=0 \text{ when } r>a. \end{aligned}\right\} \quad (3)$$

In this case, from Eqs. (1)-(3) after simple transformations we obtain

$$R_1=1-\exp\left(-\frac{a^2}{2\sigma^2}\right). \quad (4)$$

i.e., the probability of target damage is equal to the probability that the projectile will hit a circle of radius a (see [82], pages 122 and 507).

In actual practice the damage probability generally has the form shown in Fig. 3.3.2. At some distance a from the target the damage probability $G(r) = 1$, and then it diminishes to zero with an increase in r. The quantity a may here be referred to as the radius of continuous (100%) damage.

The curve shown in Fig. 3.3.2 can be approximated by various analytical expressions. For brevity we will consider only one of these expressions, and namely

$$\left.\begin{aligned} G(r) &= 1 && \text{when } r \leqslant a, \\ G(r) &= \exp[-k(r^2 - a^2)] && \text{when } r \geqslant a. \end{aligned}\right\} \tag{5}$$

From Eqs. (1), (2) and (5) after integration and simple transformations we obtain

$$R_1 = 1 - \frac{2k\sigma^2}{1 + 2k\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right). \tag{6}$$

We now examine the case in which there is a systematic firing error h. In this case, instead of Eq. (1) we will have

$$\varphi(r) = \frac{r}{\sigma^2} J_0\left(\frac{hr}{\sigma^2}\right) \exp\left(-\frac{r^2 + h^2}{2\sigma^2}\right), \tag{7}$$

where J_0 is the Bessel function (see [82], page 123).

We recall that we have the relationship

$$\int_0^\infty \varphi(r)\, dr = 1.$$

Hence, from Eq. (7) we obtain

$$\int_0^\infty r J_0\left(\frac{hr}{\sigma^2}\right) \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = \sigma^2 \exp\left(\frac{h^2}{2\sigma^2}\right). \tag{8}$$

We introduce the denotations

$$k_1 = \frac{1}{2\sigma^2}, \tag{9}$$

$$k_2 = \frac{h}{\sigma^2}. \tag{10}$$

Then, from Eq. (8) we will have

$$\int_0^\infty r J_0(k_2 r) \exp(-k_1 r^2)\, dr = \frac{1}{2k_1} \exp\left(\frac{k_2^2}{4k_1}\right). \tag{11}$$

Let us also examine the integral

$$F\left(\frac{r}{\sigma},\ \frac{h}{\sigma}\right)=\int_0^r \frac{r}{\sigma^2} J_0\left(\frac{hr}{\sigma^2}\right)\exp\left(-\frac{r^2+h^2}{2\sigma^2}\right)dr. \tag{12}$$

This integral represents the probability of hitting a circle of radius r (see Chapter 1). This probability can be found from Table 9 of the appendix. Using the denotations of (9) and (10), we can rewrite Eq. (12) to the form

$$\int_0^a rJ_0(k_2 r)\exp(-k_1 r^2)\,dr=$$
$$=\frac{1}{2k_1}\exp\left(\frac{k_2^2}{4k_1}\right)F\left(a\sqrt{2k_1},\ \frac{k_2}{\sqrt{2k_1}}\right). \tag{13}$$

We now find the probability of target damage for the damage probability (5) and the error function (7). Using Eqs. (2), (5) and (7), we find

$$R_1=\int_0^a\varphi(r)\,dr+\int_a^\infty e^{-k(r^2-a^2)}\varphi(r)\,dr=$$
$$=F\left(\frac{a}{\sigma},\ \frac{h}{\sigma}\right)+\int_0^\infty e^{-k(r^2-a^2)}\varphi(r)\,dr-\int_0^a e^{-k(r^2-a^2)}\varphi(r)\,dr. \tag{14}$$

Let us first examine the first integral in the right-hand part of Eq. (14). Using Eq. (7), we obtain

$$\int_0^\infty e^{-k(r^2-a^2)}\varphi(r)\,dr=\frac{1}{\sigma^2}\exp\left(ka^2-\frac{h^2}{2\sigma^2}\right)\times$$
$$\times\int_0^\infty rJ_0\left(\frac{hr}{\sigma^2}\right)\exp\left(-\frac{r^2}{2\sigma^2}-kr^2\right)dr. \tag{15}$$

Here we introduce the denotations

$$k_1=\frac{1}{2\sigma^2}+k, \tag{16}$$

$$k_2=\frac{h}{\sigma^2}. \tag{17}$$

The integral in the right-hand part of Eq. (15) then reduces to Expression (11) and we obtain

$$\int_0^\infty e^{-k(r^2-a^2)}\varphi(r)\,dr=$$
$$=\frac{1}{\sigma^2}\exp\left(ka^2-\frac{h^2}{2\sigma^2}\right)\frac{1}{2k_1}\exp\left(\frac{k_2^2}{4k_1}\right)=$$
$$=\frac{1}{\gamma^2}\exp\left(-\frac{kh^2}{\gamma^2}+ka^2\right), \tag{18}$$

where

$$\gamma^2=1+2k\sigma^2. \tag{19}$$

We now examine the last integral in Eq. (14). Using Eqs. (7),

(13), (16) and (17), we obtain

$$\int_0^a e^{-k(r^2-a^2)}\varphi(r)\,dr=\frac{1}{\sigma^2}\exp\left(ka^2-\frac{h^2}{2\sigma^2}\right)\times$$
$$\times\frac{1}{2K_1}\exp\left(\frac{k_2^2}{4k_1}\right)F\left(a\sqrt{2k_1},\ \frac{k_2}{\sqrt{2k_1}}\right)=$$
$$=\frac{1}{\gamma^2}F\left(\frac{a\gamma}{\sigma},\ \frac{h}{\gamma\sigma}\right)\exp\left[-\frac{kh^2}{\gamma^2}+ka^2\right]. \tag{20}$$

Finally, from Eqs. (14), (18) and (20) we find

$$R_1=F\left(\frac{a}{\sigma},\ \frac{h}{\sigma}\right)+\frac{1}{\gamma^2}\exp\left[-\frac{kh^2}{\gamma^2}+ka^2\right]\times$$
$$\times\left[1-F\left(\frac{a\gamma}{\sigma},\ \frac{h}{\gamma\sigma}\right)\right]. \tag{21}$$

Equation (21) and Table 9 of the appendix permit evaluation of the effect on firing effectiveness of the following factors: random and systematic errors (σ and h) and the parameters of the damage probability (a and k).

For greater clarity in Eq. (21) we replace the parameter k with another parameter of clearer physical significance. Into the consideration we now introduce the radius r_0 on which the damage probability $G(r_0)$ = 0.05 (i.e., so small that it can be neglected). From Eq. (5) we then find

$$k(r_0^2-a^2)=3. \tag{22}$$

We introduce the denotation

$$z=\frac{r_0}{a}. \tag{23}$$

From Eq. (22) we then obtain

$$k=\frac{3}{a^2(z^2-1)}. \tag{24}$$

From Eqs. (19) and (24) we obtain

$$\gamma^2=1+\frac{\sigma^2}{a^2}\cdot\frac{6}{z^2-1}. \tag{25}$$

Equation (21) assumes the form

$$R_1=F\left(\frac{a}{\sigma},\ \frac{h}{\sigma}\right)+\frac{1}{\gamma^2}\exp\left[\frac{3}{z^2-1}\left(1-\frac{h^2}{a^2\gamma^2}\right)\right]\times$$
$$\times\left[1-F\left(\frac{a\gamma}{\sigma},\ \frac{h}{\gamma\sigma}\right)\right]. \tag{26}$$

We see from Eqs. (25) and (26) that R_1 is a function of three arguments: a/σ, h/σ and z. Equation (26) is conveniently utilized when

$$a\neq 0 \text{ and } z\neq 1.$$

In the special case in which $a = 0$, from Eq. (22) we obtain

$$k = \frac{3}{r_0^2}, \tag{27}$$

and Eq. (21) assumes the form

$$R_1 = \frac{r_0^2}{r_0^2 + 6\sigma^2} \exp\left[-\frac{3h^2}{r_0^2 + 6\sigma^2}\right]. \tag{28}$$

When $a \neq 0$, $z = 1$, Eq. (21) remains only with the single first term

$$R_1 = F\left(\frac{a}{\sigma}, \frac{h}{\sigma}\right). \tag{29}$$

In the special case in which $h = 0$, from Eq. (21) we obtain Eq. (6) which may be rewritten to the form

$$R_1 = 1 - \frac{6\sigma^2}{6\sigma^2 + a^2(z^2 - 1)} \exp\left(-\frac{a^2}{2\sigma^2}\right). \tag{30}$$

Equations (26), (28) and (30) make it possible to analyze the effect of various factors on the effectiveness of firing at a point target.

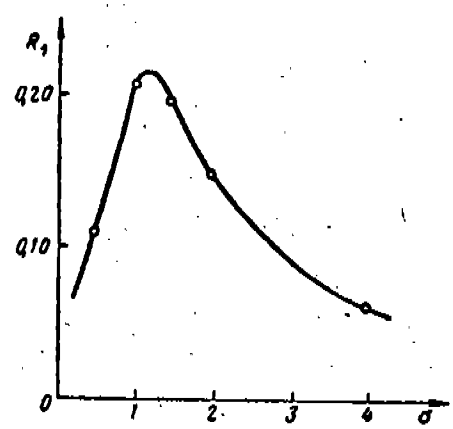

Fig. 3.3.3

EXAMPLE. Given $a = 1$, $z = 2$, $h = 2$. Find R_1 as a function of σ.

Solution. From Eqs. (25) and (26) we obtain

$$\gamma^2 = 1 + 2\sigma^2,$$

$$R_1 = F\left(\frac{1}{\sigma}, \frac{2}{\sigma}\right) + \frac{1}{\gamma^2} \exp\left[1 - \frac{4}{\gamma^2}\right]\left[1 - F\left(\frac{\gamma}{\sigma}, \frac{2}{\gamma\sigma}\right)\right].$$

Having assumed various values of σ, by means of these equations and Table 9 of the appendix we find R_1 as a function of σ as shown in Fig. 3.3.3. We see from this figure that there exists

a value σ ($\cong 1$) which corresponds to the maximum of the damage probability R_1.

Hence it follows that with systematic errors it occasionally makes sense to introduce the artificial scattering which leads to an increase in the probability of target damage. Thus, if under the conditions of the subject example $\sigma = 0.5$, to increase the probability of target damage σ should be approximately doubled.

§3.4. EVALUATION OF FIRING EFFECTIVENESS IN THE CASE OF INDEPENDENT SHOTS

A. Determination of Probability for at Least a Single Hit on the Target for n Independent Shots

Let n independent shots be fired at a target, with the probability of hitting the target for each of these shots being identical and equal to p. In this case the probability of missing the target with a single shot will be

$$q = 1 - p. \tag{1}$$

The probability that in n shots there will be not a single hit is equal to

$$q^n = (1 - p)^n. \tag{2}$$

Hence the probability of at least a single hit on the target will be

$$P'_n = 1 - (1 - p)^n. \tag{3}$$

With large n utilization of Eq. (3) becomes inconvenient for calculations. In this case, Eq. (3) can be replaced by an approximate equation in the following manner – it may be rewritten to the form:

$$P'_n = 1 - \left(1 - \frac{np}{n}\right)^n. \tag{4}$$

Since

$$\lim_{n \to \infty} \left(1 - \frac{a}{n}\right)^n = e^{-a}, \tag{5}$$

Eq. (4) may be written approximately as

$$P'_n = 1 - e^{-np}. \tag{6}$$

The advantage of Eq. (6) over Eq. (3) is that for Eq. (3) we need a table with two inputs, while for Eq. (6) it is enough to have a table with a single input.

To evaluate the accuracy of Eq. (6) we find the ratio

$$k = \frac{1 - (1 - p)^n}{1 - e^{-np}}. \tag{7}$$

The larger n, the closer k to unity. We introduce the deno-

$$k = 1 + \frac{m}{1000}. \tag{8}$$

Table 10 of the appendix shows m as a function of n and np. We see from this table that with $n > 30$ the calculational errors due to Eq. (6) do not exceed 1%, since m does not exceed 10-11. With $n > 50$ the errors in calculation due to Eq. (6) do not exceed 0.5%.

Table 10 also shows that when $np \leq 0.10$ the error in calculation due to Eq. (6) does not exceed 1%, as soon as $n \geq 5$. Thus, calculation with Eq. (6) exhibits rather high accuracy in two cases:

1) for large n and any p;

2) for small np and any n.

From Eqs. (7) and (8) we have the formula

$$P'_n = \left(1 + \frac{m}{1000}\right)(1 - e^{-np}). \tag{9}$$

With this formula it is possible to determine P'_n for any n and p by means of Table 10.

B. Determination of the Probability of a Specific Number of Hits with Independent Shots

Let n independent shots be fired at a target, with the probability of hitting the target for each of these shots equal to p. In this case, the probability of obtaining exactly m hits will be (see [82], page 54)

$$P_m = C_n^m p^m q^{n-m}, \tag{10}$$

where

$$q = 1 - p. \tag{11}$$

C. The Number of Shots Prior to Achieving a Single Hit on the Target

Let us examine the case of firing separate independent shots with a constant probability p of achieving a hit with a single shot. Let the result of each shot be observed until the following shot is fired, with the firing stopped as soon as the first hit on the target is achieved. Under these conditions, the number N of shots prior to hitting the target is a random quantity. Let us find the distribution moments of this random quantity. According to definition, for the mathematical expectation we have

$$M(N) = 1P_1 + 2P_2 + 3P_3 + \ldots, \tag{12}$$

where P_i is the probability of achieving a hit with the ith shot under the condition that it has not been achieved in any of the previous shots.

It is obvious that

$$P_i = pq^{i-1} = p(1-p)^{i-1}. \qquad (13)$$

From Eqs. (12) and (13) we find

$$M(N) = p + 2pq + 3pq^2 + \ldots = \\ = p(1 + 2q + 3q^2 + \ldots) = p\frac{1}{(1-q)^2} = \frac{1}{p}. \qquad (14)$$

D. The Number of Shots to Achieve a Given Probability for at Least a Single Hit on the Target

Let single shots with the probability p of hitting the target with a single shot be fired at a target individually. The probability of at least one hit on the target is found from Eq. (3).

Given the probability α for at least one hit on a target, Eq. (3) assumes the form

$$\alpha = 1 - (1-p)^{N_\alpha}. \qquad (15)$$

Hence the needed number N_α of shots is found in the form

$$N_\alpha = \frac{\ln(1-\alpha)}{\ln(1-p)}. \qquad (16)$$

The approximate formula for N_α can be found from Eq. (6)

$$\alpha = 1 - e^{-pN_\alpha}, \qquad (17)$$

whence

$$N_\alpha \cong \frac{C_\alpha}{p}, \qquad (18)$$

where

$$C_\alpha = |\ln(1-\alpha)|. \qquad (19)$$

Below we give C_α as a function of α according to Eq. (19)

α, %	50	60	70	80	90	95	99
C_α	0.69	0.92	1.20	1.61	2.30	3.00	4.61

Equation (18), on the basis of Eq. (14), can be rewritten to the form

$$N_\alpha = C_\alpha N_{cp} = C_\alpha M(N). \qquad (20)$$

E. Determination of Target Damage Probability

In the previous items of this section we spoke of the probability of hitting a target. Here we will consider the problem of determining the unconditional probability of target damage, if we know the damage probability G_m.

Let us consider the case in which n shots are fired at a target. Let P_m denote the probability of m hits occurring during

these n shots. From the familiar formula for the total probability we will then find that the probability of target damage with n shots will be

$$R_n = P_1G_1 + P_2G_2 + \ldots + P_nG_n = \sum_{i=0}^{n} P_iG_i. \tag{21}$$

It is obvious that with an increase in i the probabilities P_i diminish. On the other hand, the probabilities G_i increase with greater i, which calls for a sufficiently large number of terms for large n to be taken in Eq. (21). With large n it is therefore more convenient to transform Eq. (21) so that all terms in that equation diminish with increasing i.

For this we will write the obvious equality

$$1 = \sum_{i=0}^{n} P_i. \tag{22}$$

Subtracting Eq. (21) from Eq. (22), we obtain

$$\begin{aligned} 1 - R_n = \overline{R_n} = \sum_{i=0}^{n} P_i(1 - G_i) = \sum_{i=0}^{n} P_i\overline{G}_i = \\ = P_0 + \sum_{i=1}^{n} P_i\overline{G}_i, \end{aligned} \tag{23}$$

where for brevity we introduced the denotation

$$\overline{G}_i = 1 - G_i.$$

In Eq. (23) with increasing i, the terms in the right-hand portion diminish, thus making it possible in the calculations to limit ourselves frequently to a small number of terms. Moreover, this equation is of great fundamental significance — it demonstrates that the greatest role in damage probability R_n is played by the first probabilities of the damage probability. We clarify this with an example.

EXAMPLE 1. Let the probability of hitting a target with a single shot be equal to 0.1. Up to 3 independent shots are fired at the target. The target damage probability is characterized by the probability $G_1 = 0.5$. The probabilities G_2 and G_3 are unknown. Evaluate the damage probability R_3.

Solution. We find the probabilities of none, of one, of two and of three hits:

$$P_0 = (0{,}9)^3 = 0{,}729,$$
$$P_1 = 3 \cdot 0{,}1 \cdot (0{,}9)^2 = 0.243,$$
$$P_2 = 3(0{,}1)^2 \cdot 0{,}9 = 0{,}027,$$
$$P_3 = (0{,}1)^3 = 0{,}001.$$

Let us examine three versions of the damage probability.

Ist version: $G_2 = G_3 = 1$. With Eq. (23) we find

$$\bar{R}_3 = 0,729 + 0,243 \cdot 0,5 = 0,850.$$

IInd version: the damage probability is exponential. Then

$$\bar{G}_2 = (0,5)^2 = 0,25, \quad \bar{G}_3 = (0.5)^3 = 0,125$$

and from Eq. (23) we find

$$R_3 = 0,729 + 0,243 \cdot 0,5 + 0,027 \cdot 0,25 + 0,001 \cdot 0,125 = 0,857.$$

IIIrd version: $G_2 = G_3 = 0.5$. From Eq. (23) we find

$$\bar{R} = 0,729 + 0,243 \cdot 0,5 + 0,027 \cdot 0.5 + 0,001 \cdot 0,5 = 0,864.$$

Hence we see how little the quantities G_2 and G_3 affect the damage probability R_3. We also note that versions I and III are the extreme possible versions, while the exponential damage probability occupies an intermediate position between the extreme possibilities.

F. Calculation of Damage Probability for Independent Shots and an Exponential Target Damage Probability

Let us consider the case of an exponential target damage probability. In this case we have

$$\bar{G}_i = \bar{G}_1^i. \tag{24}$$

From Eqs. (23), (24) and (10) we find

$$\bar{R}_n = \sum_{i=0}^{n} C_n^i (p\bar{G}_1)^i q^{n-i} = (p\bar{G}_1 + q)^n. \tag{25}$$

In Eq. (25), having substituted

$$\bar{G}_1 = 1 - \frac{1}{\omega},$$

we will obtain

$$\bar{R}_n = \left[p\left(1 - \frac{1}{\omega}\right) + 1 - p\right]^n = \left(1 - \frac{p}{\omega}\right)^n,$$

whence

$$R_n = 1 - \left(1 - \frac{p}{\omega}\right)^n. \tag{26}$$

Comparing Eq. (26) with Eq. (3) we see that with an exponential damage probability the probability of damaging a given target with n shots is equal to the probability of at least one hit in the reference target for which the hit probability with a single shot is smaller by a factor of ω than in the given target.

From Eqs. (26) and (6) we obtain

$$R_n = 1 - e^{-\frac{np}{\omega}}. \tag{27}$$

G. Calculating the Number of Shots Required to Damage the Target

First we will consider the general case of the target damage probability. In this case the average number of shots to target damage will be

$$M(N) = 1P_1 + 2P_2 + 3P_3 + \ldots, \tag{28}$$

where P_i is the probability of damaging the target on the <u>*i*th</u> shot given the condition that the target was not damaged with any of the previous shots. Obviously,

$$P_i = R_i - R_{i-1}, \tag{29}$$

where R_i is the probability of damaging the target with i shots.

From Eqs. (28) and (29) we obtain

$$M(N) = 1R_1 + 2(R_2 - R_1) + 3(R_3 - R_2) + \ldots + \\ + n(R_n - R_{n-1}) + \ldots \tag{30}$$

Since with $n \to \infty$ we have $R_n \to 1$, beginning with some value of n we can assume $R_n = 1$ and $R_{n+1} - R_n = 0$. Equation (30) may then be written as follows:

$$M(N) \cong -R_1 - R_2 - \ldots - R_{n-1} + n = \\ = (1 - R_0) + (1 - R_1) + \ldots + (1 - R_{n-1}), \tag{31}$$

where $R_0 = 0$.

The larger n in Eq. (31), the more exact this equation. Instead of the approximate equation (31) we can therefore write the exact equation

$$M(N) = \sum_{i=0}^{\infty} (1 - R_i). \tag{32}$$

In the special case of the exponential target damage probability we find from Eqs. (32) and (26), after simple transformations

$$M(N) = N_{cp} = \frac{\omega}{p}. \tag{33}$$

Obviously, Eq. (20) remains valid even if N_{sr} is determined from Eq. (33).

H. Calculation of the Mathematical Expectation for the Number of Damaged Targets

Let n shots be fired at a group of k targets with the damage probability for the <u>*i*th</u> target during this firing equal to R_i. In this case, the number of targets which are damaged during the firing operation is random. Let us find the mathematical ex-

pectation of that number. We introduce the random quantity x_i which is equal to 1 for the damaging of the ith target and equal to zero if this target is not damaged. The probability of the equality $x_i = 1$ is R_i, while the probability of the equality $x_i = 0$ is equal to $1 - R_i$.

The number of targets damaged during the firing operation will be

$$x = x_1 + x_2 + \dots + x_k,$$

and its mathematical expectation

$$M(x) = \sum_{i=1}^{k} M(x_i) = \sum_{i=1}^{k} [R_i + 0(1 - R_i)] = \sum_{i=1}^{k} R_i. \qquad (34)$$

Thus the mathematical expectation of the number of damaged targets is equal to the sum of the damage probabilities for these targets. We note that this is valid for any relationship between the shots.

Let us now consider the special case of independent shots and $\omega = 1$ (i.e., to damage the target, a single hit is enough).

Let there be k targets of identical area and let the probability of hitting each of these targets with each shot be identical and equal to p/k, where p is the probability of hitting the target with a single shot (regardless of which target). We denote

$$a = \frac{n}{k} \qquad (35)$$

as the average firing density, i.e., the number of shots fired during the firing operation at a single target.

The probability R_i of damaging the ith target is found to be the probability of hitting this target at least once

$$R_i = 1 - \left(1 - \frac{p}{k}\right)^n. \qquad (36)$$

Using Eq. (35), we can write

$$R_i = 1 - \left[\left(1 - \frac{p}{k}\right)^k\right]^a. \qquad (37)$$

We have approximately

$$\left(1 - \frac{p}{k}\right)^k = e^{-p}, \qquad (38)$$

whence

$$R_i = 1 - e^{-ap}. \qquad (39)$$

According to Eq. (34) we find the mathematical expectation of the number of damaged targets

$$x_{cp} = kR_i = k(1 - e^{-ap}). \quad (40)$$

Hence the mean number of damaged targets (in %) will be

$$x_{cp}\% = \frac{x_{cp} \cdot 100}{k} = 100(1 - e^{-ap}). \quad (41)$$

Table 3.4.1 shows ap as a function of x_{sr}% from Eq. (41).

TABLE 3.4.1

x_{cp}, %	ap	x_{cp}, %	ap
5	0.051	60	0.916
10	0.105	70	1.204
20	0.223	80	1.609
30	0.357	90	2.303
40	0.511	95	2.996
50	0.693	99	4.605

EXAMPLE 2. Let firing be conducted at 50 targets under the condition that the probability of hitting any of the targets is equal to 0.5 with a single shot. What number of shots in this firing operation corresponds to the mathematical expectation of damaging 50% of the targets?

Solution. From Table 3.4.1 we find $ap = 0.693$, whence $a = (0.693/0.5) = 1.38$. Further, from Eq. (35) we find

$$n = ak = 1.38 \cdot 50 = 69.$$

§3.5. EVALUATION OF FIRING EFFECTIVENESS IN THE CASE OF DEPENDENT SHOTS

(The case of a scheme of two error groups)

A. The Scheme of Two Firing Error Groups

The scheme of two error groups is extensively employed in practice. Moreover, the calculational formulas are simplified here. We will therefore limit ourselves to an examination of this simple case. For brevity we will first consider firing operations with one-dimensional scattering along the horizontal axis (Fig. 3.5.1).

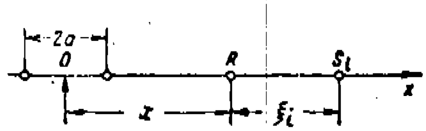

Fig. 3.5.1

Let the center of the target be situated at point O. Let the trajectory for the ith shot pass through the point S_i ($i = 1, 2, \ldots, n$), and let the center of the trajectory scattering for a group consisting of n shots be situated at point R. We denote the deviation of the ith trajectory from point R by ξ_i:

$$\xi_i = RS_i. \quad (1)$$

The deviation of the point R from the center of the target is denoted x:

$$x = OR. \tag{2}$$

The deviation of the trajectory from the target will then be

$$z_i = \xi_i + x. \tag{3}$$

The quantities ξ_i are distributed normally with the mathematical expectation 0 and the standard deviation σ_n which characterizes the technical scattering of the shots. The quantities ξ_i are obviously independent of each other and are known as nonrecurring errors.

The quantity x is constant for a group consisting of n shots but varies randomly with transition from one group to another. The distribution of the quantity x is normal with the mathematical expectation x_0 and with the standard deviation σ_g. The quantity σ_g is the characteristic for the scattering of group recurring errors which in practice are generally errors in firing preparation. It is obvious that these quantities x and ξ_i are independent of each other.

Let us examine the total firing errors z_i. These are no longer independent. For these we have:

$$M(z_i) = M(\xi_i) + M(x) = 0 + x_0 = x_0, \tag{4}$$

$$\sigma^2(z_i) = \sigma^2(\xi_i) + \sigma^2(x) = \sigma_{\text{н}}^2 + \sigma_{\text{г}}^2, \tag{5}$$

$$\begin{aligned} K_{ij} &= M(z_i z_j) - M(z_i)\,M(z_j) = \\ &= M(\xi_i\xi_j + \xi_i x + \xi_j x + x^2) - x_0^2 = \\ &= 0 + x_0 M(\xi_i) + x_0 M(\xi_j) + \sigma_{\text{г}}^2 = \sigma_{\text{г}}^2. \end{aligned} \tag{6}$$

The correlation factor between quantities z_i and z_j is found from the equation

$$r_{ij} = \frac{K_{ij}}{\sigma(z_i)\sigma(z_j)} = \frac{\sigma_{\text{г}}^2}{\sigma^2(z)} = \frac{\sigma_{\text{г}}^2}{\sigma_{\text{н}}^2 + \sigma_{\text{г}}^2}. \tag{7}$$

Equations (1)-(7) characterize the so-called scheme of two groups of firing errors (compare [82], page 168).

B. Calculation of the Probability of Hitting a Target in the Case of a Scheme of Two Groups of Firing Errors with One-Dimensional Scattering

Let us examine the case in which the target has a width $2a$ and in which its center coincides with the coordinate origin (Fig. 3.5.1).

The error distribution densities for the first and second

groups will be

$$\varphi_{\text{н}}(\xi) = \frac{1}{\sigma_{\text{н}}\sqrt{2\pi}} e^{-\frac{(\xi-x)^2}{2\sigma_{\text{н}}^2}} = \frac{1}{\sigma_{\text{н}}} \varphi_0\left(\frac{\xi - x}{\sigma_{\text{н}}}\right), \tag{8}$$

$$\varphi_{\text{г}}(x) = \frac{1}{\sigma_{\text{г}}\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma_{\text{г}}^2}} = \frac{1}{\sigma_{\text{г}}} \varphi_0\left(\frac{x - x_0}{\sigma_{\text{г}}}\right), \tag{9}$$

where

$$\varphi_0(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}. \tag{10}$$

With a fixed error x the conditional probability of hitting the target with a single shot (under the condition that the group error is equal to x) will be

$$P(x) = \int_{-a}^{a} \frac{1}{\sigma_{\text{н}}} \varphi_0\left(\frac{\xi - x}{\sigma_{\text{н}}}\right) d\xi =$$

$$= \frac{1}{2}\left[\Phi\left(\frac{a - x}{\sigma_{\text{н}}}\right) + \Phi\left(\frac{a + x}{\sigma_{\text{н}}}\right)\right], \tag{11}$$

where $\Phi(u)$ is the Laplace function.

The unconditional probability of hitting the target with a single shot will be

$$P_1 = \int_{-\infty}^{+\infty} P(x)\varphi_{\text{г}}(x)\,dx =$$

$$= \int_{-\infty}^{+\infty}\int_{-a}^{a} \frac{1}{\sigma_{\text{н}}}\varphi_0\left(\frac{\xi - x}{\sigma_{\text{н}}}\right)\frac{1}{\sigma_{\text{г}}}\varphi_0\left(\frac{x - x_0}{\sigma_{\text{г}}}\right) dx\,d\xi =$$

$$= \int_{-a}^{a} d\xi \int_{-\infty}^{+\infty} \frac{1}{\sigma_{\text{н}}}\varphi_0\left(\frac{\xi - x}{\sigma_{\text{н}}}\right)\frac{1}{\sigma_{\text{г}}}\varphi_0\left(\frac{x - x_0}{\sigma_{\text{г}}}\right) dx. \tag{12}$$

The second integral in Eq. (12) represents a composition of two normal distribution. As a result of this composition, we obtain a normal distribution with the mathematical expectation x_0 and the dispersion

$$\sigma_{\Sigma}^2 = \sigma_{\text{н}}^2 + \sigma_{\text{г}}^2 \tag{13}$$

(see [82], page 68). Equation (12) may therefore be transformed in the following manner:

$$P_1 = \int_{-a}^{a} \frac{1}{\sigma_{\Sigma}}\varphi_0\left(\frac{\xi - x_0}{\sigma_{\Sigma}}\right) d\xi =$$

$$= \frac{1}{2}\left[\Phi\left(\frac{a - x_0}{\sigma_{\Sigma}}\right) + \Phi\left(\frac{a + x_0}{\sigma_{\Sigma}}\right)\right]. \tag{14}$$

Hence it follows that in a scheme of two error groups the

hit probability with a single shot is determined in the same manner as in the case of a single error group, however, with replacement of the technical scattering by the total scattering of the two error groups.

The probability of at least one hit on the target with n shots is found from the equation

$$P'_n = \int_{-\infty}^{+\infty} \{1 - [1 - P(x)]^n\} \frac{1}{\sigma_r} \varphi_r\left(\frac{x - x_0}{\sigma_r}\right) dx, \qquad (15)$$

where $P(x)$ is found from Eq. (11).

The integral in Eq. (15) is easily calculated by one of the methods of numerical integration (for example, by the Simpson method).

If the target damage probability is exponential, from Eq. (15) we obtain the target damage probability in firing a group of n shots

$$R_n = \int_{-\infty}^{+\infty} \left\{1 - \left[1 - \frac{P(x)}{\omega}\right]^n\right\} \frac{1}{\sigma_r} \varphi_0\left(\frac{x - x_0}{\sigma_r}\right) dx. \qquad (16)$$

C. Determination of Target Hit Probability for a Scheme of Two Firing Error Groups with Two-Dimensional Scattering

For simplicity let us consider the case of firing under the following conditions:

a) the target is a square with a side $2a$;

b) the shot scattering is circular;

c) there are no systematic firing errors.

Let the coordinate origin coincide with the target center. We denote the errors of the first group (nonrecurring) by ξ and η, and we denote the errors of the second group (recurring) by x and y.

The probability density for the errors of the 1st and 2nd groups will then be, respectively,

$$\frac{1}{\sigma_{\text{II}}^2} \varphi_0\left(\frac{\xi - x}{\sigma_{\text{II}}}\right) \varphi_0\left(\frac{\eta - y}{\sigma_{\text{II}}}\right), \qquad (17)$$

$$\frac{1}{\sigma_r^2} \varphi_0\left(\frac{x}{\sigma_r}\right) \varphi_0\left(\frac{y}{\sigma_r}\right). \qquad (18)$$

With fixed magnitudes for the errors of the second group, the probability of hitting the target with a single shot will be

$$P(x, y) = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{\sigma_{\text{н}}^2} \varphi_0\left(\frac{\xi - x}{\sigma_{\text{н}}}\right) \varphi_0\left(\frac{\eta - y}{\sigma_{\text{н}}}\right) d\xi\, d\eta, \tag{19}$$

whence we find

$$P(x, y) = \frac{1}{4}\left[\Phi\left(\frac{a - x}{\sigma_{\text{н}}}\right) + \Phi\left(\frac{a + x}{\sigma_{\text{н}}}\right)\right] \times$$
$$\times \left[\Phi\left(\frac{a - y}{\sigma_{\text{н}}}\right) + \Phi\left(\frac{a + y}{\sigma_{\text{н}}}\right)\right]. \tag{20}$$

The conditional probability of achieving m hits in the target out of n shots (with fixed x and y) will be

$$P_{m,n}(x, y) = C_n^m [P(x, y)]^m [1 - P(x, y)]^{n-m}, \tag{21}$$

where C_n^m is the number of combinations of n items taken m at a time.

The unconditional probability of achieving m hits from n shots will be

$$P_{m,n} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_{m,n}(x, y) \frac{1}{\sigma_r^2} \varphi_0\left(\frac{x}{\sigma_r}\right) \varphi_0\left(\frac{y}{\sigma_r}\right) dx\, dy. \tag{22}$$

Integral (22) is simply calculated in the special case in which $n = m = 1$. In this case we obtain the probability of hitting the target with a single shot, which, as follows from Eqs. (12)-(14), can be written in the following form:

$$P_{11} = \left[\Phi\left(\frac{a}{\sigma_\Sigma}\right)\right]^2, \tag{23}$$

where σ_Σ is found from Eq. (13).

When $n > 1$ Integral (22) can be calculated by one of the methods of numerical integration. In this case it reduces to the product of two single integrals. We will demonstrate how this is done in the special case of $n = 2$.

Let us first consider the probability P_{22}. From Eqs. (22), (21) and (20) we find

$$P_{22} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [P(x, y)]^2 \frac{1}{\sigma_r^2} \varphi_0\left(\frac{x}{\sigma_r}\right) \varphi_0\left(\frac{y}{\sigma_r}\right) dx\, dy =$$
$$= \int_{-\infty}^{+\infty} \frac{1}{4}\left[\Phi\left(\frac{a - x}{\sigma_{\text{н}}}\right) + \Phi\left(\frac{a + x}{\sigma_{\text{н}}}\right)\right]^2 \frac{1}{\sigma_r} \varphi_0\left(\frac{x}{\sigma_r}\right) dx \times$$
$$\times \int_{-\infty}^{+\infty} \frac{1}{4}\left[\Phi\left(\frac{a - y}{\sigma_{\text{н}}}\right) + \Phi\left(\frac{a + y}{\sigma_{\text{н}}}\right)\right]^2 \frac{1}{\sigma_r} \varphi_0\left(\frac{y}{\sigma_r}\right) dy = A^2. \tag{24}$$

In Eq. (24) we obtained the product of two identical single integrals, each of which we denote A. In view of the integrand

symmetry we can write the following expression for A:

$$A=\frac{1}{2}\int_{0}^{\infty}\left[\Phi\left(\frac{a-x}{\sigma_n}\right)+\Phi\left(\frac{a+x}{\sigma_n}\right)\right]^2\frac{1}{\sigma_r}\varphi_0\left(\frac{x}{\sigma_r}\right)dx=$$

$$=\frac{1}{2}\int_{0}^{\infty}\left[\Phi\left(\frac{a}{\sigma_n}-\varepsilon z\right)+\Phi\left(\frac{a}{\sigma_n}+\varepsilon z\right)\right]^2\varphi_0(z)\,dz. \qquad (25)$$

In Eq. (25) we have substituted $z = x/\sigma_g$ and introduced the denotation

$$\varepsilon=\frac{\sigma_r}{\sigma_n}. \qquad (26)$$

We note that the magnitude of A is fully defined by two parameters: ε and P_{11}.

Indeed, from Eqs. (26) and (13) we find

$$\sigma_\Sigma^2=\sigma_n^2(1+\varepsilon^2), \qquad (27)$$

whence

$$\frac{a}{\sigma_n}=\frac{a}{\sigma_\Sigma}\sqrt{1+\varepsilon^2}. \qquad (28)$$

If P_{11} and ε are given, from Eq. (23) we determine the ratio a/σ_Σ, while from Eq. (27) we determine the ratio a/σ_n which is included in Eq. (25) for A.

We now examine the probability P_{12}. From Eqs. (22), (21) and (20) we find

$$P_{12}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}2P(x,y)\left[1-P(x,y)\right]\frac{1}{\sigma_r^2}\varphi_0\left(\frac{x}{\sigma_r}\right)\varphi_0\left(\frac{y}{\sigma_r}\right)dx\,dy=$$

$$=2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}P(x,y)\frac{1}{\sigma_r^2}\varphi_0\left(\frac{x}{\sigma_r}\right)\varphi_0\left(\frac{y}{\sigma_r}\right)dx\,dy-2P_{22}. \qquad (29)$$

However, the integral in the right-hand portion of Eq. (29) is equal to P_{11}. We therefore obtain

$$P_{12}=2P_{11}-2P_{22}=2P_{11}-2A^2. \qquad (30)$$

Now it is not difficult to find the last of the probabilities P_{02} of interest to us. For this we have to use the fact that the sum of the probabilities P_{02}, P_{12} and P_{22} is equal to 1. From Eqs. (30) and (24) we then obtain

$$P_{02}=1-2P_{11}+A^2. \qquad (13)$$

We see from Eqs. (24), (30) and (31) that to calculate all three probabilities of interest to us we need only one single integration to calculate .. according to Eq. (25).

EXAMPLE I. Let us consider the case in which the target is a

square having a side $2a$ = 4 m, and the standard firing errors for the two groups are identical: $\sigma_n = \sigma_g$ = 1 m. Under these conditions, let us find the probabilities P_{02}, P_{12} and P_{22}.

Solution. From Eq. (25), by numerical integration, we find A = 0.732. From Eq. (13) we obtain $\sigma_\Sigma = (2)^{-2}$, while from Eq. (23) we find P_{11} = 0.710.

Further, from Eqs. (24), (29) and (30), we obtain P_{02} = 0.116, P_{12} = 0.348, P_{22} = 0.536.

We note that in the case of independent shots with P_{11} = = 0.710 we will have

$$P_{02}=(1-P_{11})^2=0{,}084,$$
$$P_{12}=2P_{11}(1-P_{11})=0{,}412,$$
$$P_{22}=P_{11}^2=0{,}504.$$

The results of these calculations are summarized in Table 3.5.1.

TABLE 3.5.1

ε	0	1	∞
p_{02}	0,084	0,116	0,290
p_{12}	0,412	0,348	0
p_{22}	0,504	0,536	0,710
p'	0,916	0,884	0,710
r	0	0,5	1

Table 3.5.1 also shows the values of the probability P' for at least a single hit on the target

$$P'=P_{12}+P_{22},$$

as well as the values for the correlation coefficient

$$r=\frac{\sigma_r^2}{\sigma_n^2+\sigma_r^2}=\frac{\varepsilon^2}{1+\varepsilon^2}. \qquad (32)$$

Moreover, Table 3.5.1 shows the probability values for ε = = ∞ (r = 1). Here the nonrecurring errors are so small that the technical scattering of the shots can be neglected and it may be assumed that all of the shots in one group hit one point. This means that if one shot hits the target, all of the remaining shots would also hit the target. In this case therefore P_{12} = 0, $P_{22} = P_{11}$ and $P_{02} = 1 - P_{11}$.

The quantitative relationships seen in an examination of Table 3.5.1 are general. With an increase in ε (when $\varepsilon \to \infty$ or $r \to$ $\to 1$) the following quantitative relationships prevail for any values of n:

1) the quantities P_{1n}, P_{2n}, ..., $P_{n-1,n}$ tend toward 0;

2) the quantity $P_{n,n}$ tends toward P_{11};

3) the quantity P_{0n} tends toward $1 - P_{11}$;

4) the probability P' of at least one hit tends toward P_{11}.

We also note that when $s \leq 1 \left(r < \frac{1}{2}\right)$ the probability characteristics do not significantly differ from those which correspond to independent shots.

D. Determination of Target Damage Probability for the Case of a Scheme of Two Error Groups

For conciseness in the discussion we will limit ourselves here to the case in which the number of shots in the group is equal to 2.

In this case, according to Eq. (3.4.21), the probability of target damage will be

$$R_2 = P_{12}G_1 + P_{22}G_2, \tag{33}$$

where the probabilities P_{12} and P_{22} are found from Eqs.(24) and (30), while the probabilities G_1 and G_2 are defined by the damage probability.

We will illustrate Eq. (33) by means of an example.

EXAMPLE 2. Find the target damage probability under the conditions of EXAMPLE 1 for three versions of the damage probability:

$$\begin{array}{ll} \text{a) } G_1 = 0.5, & G_2 = 0.5, \\ \text{b) } G_1 = 0.5, & G_2 = 0.75, \\ \text{c) } G_1 = 0.5, & G_2 = 1, \end{array}$$

(here, case b) corresponds to an exponential damage probability).

Solution. Under the conditions of EXAMPLE 1 we had $P_{12} = 0.348$ and $P_{22} = 0.536$.

From Eq. (33) we determine the values of R_2 shown in Table 3.5.2.

TABLE 3.5.2

Damage probability variant	R_2
a	0.442
b	0.576
c	0.710

We see from this table that the exponential damage probability occupies an intermediate position (with respect to R_2) between the other cases considered.

E. Application of the Method of Statistical Modeling to Determine a Target Damage Probability

We limited ourselves above to an examination of several special cases and to several examples for the determination of the target damage probability. We did this because in the more general cases we obtain extremely cumbersome analytical expressions which are of little use in calculation.

The method of statistical modeling makes it possible with a single simple general scheme to determine the target damage probability for the most general conditions:

an arbitrary target shape;

an arbitrary number of shots in the group;

noncircular technical scattering with arbitrary direction for the scattering axes relative to the target;

noncircular scattering of recurring errors with arbitrary direction for the scattering axes relative to the target;

there may be systematic errors;

the relationship between the errors of successive shots is arbitrary (not necessarily reducing to the scheme of two error groups);

the target damage probability is arbitrary.

The scheme for the application of the method of statistical tests to determine the target damage probability by a group consisting of n shots involves the following:

1) One group of n shots is modeled. By means of random-number sensors we determine the recurring and nonrecurring errors and for each shot we determine whether it is a hit or a miss. We calculate the number of hits. By means of the random-number sensor and the damage probability we determine whether the given group has damaged or failed to damage the target.

2) Modeling of the shot group is repeated N times. As a result we have that the target was damaged N_1 times, while failure to damage the target occurred $N - N_1$ times.

3) We find the target damage probability from the frequency

$$P = \frac{N_1}{N}. \tag{34}$$

4) We calculate the accuracy of the solution according to Eq. (34) by means of the method discussed in §2.3. If the accuracy is inadequate, the modeling is continued until the given accuracy is achieved.

§3.6. CONSIDERATION OF RELIABILITY IN EVALUATING FIRING EFFECTIVENESS

A. Formulation of the Problem

Let us examine the case of firing n shots at a single target under the following conditions.

1. The probability characteristics of the projectile [missile] remain unchanged from shot to shot (the probability of hitting the target, the probability of on-board facility failure, etc.).

2. The probability of no weapon (launch installation) failure with n shots is written in the form

$$P(n) = P_1^n \tag{1}$$

(see Eq. (1.8.5), where P_1 is the probability of no failure with the first shot, if there were no malfunctions prior to that shot.

3. In evaluating the target damage probability we can neglect the accumulation of damage due to the previous shots (if the target was not damaged by these shots).

Under these assumptions we can write the following expression for the unconditional target damage probability with a single shot:

$$\tilde{R}_1 = P_1 \cdot R_1, \tag{2}$$

where R_1 is the conditional probability of target damage with a single shot, determined for the condition that the weapon did not fail on that shot.

Let us now examine the following problem: the firing installation has at its disposal n projectiles, the firing is being carried out against a single target, the results of the firing are under observation and the firing is stopped when the target is damaged. We are required to determine the probability $\tilde{R}_n$ of target damage.

In analogy with §3.4.1, for the case of independent shots, this problem has the following solution [see Eq. (3.4.26)]:

$$\tilde{R}_n = 1 - (1 - \tilde{R}_1)^n. \tag{3}$$

However, under these conditions Eq. (3) is not valid, since the possibility of weapon failure makes the shots dependent. Indeed, if the weapon failed on some shot, the target damage probability for all of the subsequent shots is equal to zero.

B. Basic Equations

Let Q_k denote the probability of the following event: the target has not been damaged by shots with numbers from 1 to $k - 1$;

on the *kth* shot the target was damaged.

The quantity Q_k is easily defined as the product of three factors:

the probability of proper weapon function for k shots is P_1^k;

the probability of target nondamage with $k-1$ shots is $(1 - R_1)^{k-1}$;

the probability of target damage on the *kth* shot is R_1, i.e.,

$$Q_k = P_1^k R_1 (1 - R_1)^{k-1} = P_1 R_1 z^{k-1}, \tag{4}$$

where for brevity we have introduced the denotation

$$z = P_1 (1 - R_1). \tag{5}$$

We can now write the following obvious equations:

$$\tilde{R}_n = \sum_{k=1}^{n} Q_k, \tag{6}$$

$$M_n = \sum_{k=1}^{n} k Q_k, \tag{7}$$

where M_n is the mathematical expectation of the number of shots.

Using Eq. (4) and the relationship

$$\sum_{k=1}^{n} z^{k-1} = \frac{1 - z^n}{1 - z}, \tag{8}$$

from Eq. (6) we obtain

$$\tilde{R}_n = P_1 R_1 \frac{1 - z^n}{1 - z}. \tag{9}$$

This equation is valid when the weapon (launch installation) is clearly functioning properly at the beginning of the firing operation. If it is not known in advance whether or not the weapon (launch installation) is functioning properly, Eq. (9) should be written in the following form:

$$\tilde{R}_n = K_g P_1 R_1 \frac{1 - z^n}{1 - z}, \tag{9a}$$

where K_g is the coefficient of weapon (launch installation) readiness (see §1.9).

Equation (7) can be rewritten to the form

$$M_n = P_1 R_1 \sum_{k=1}^{n} k z^{k-1} = P_1 R_1 \frac{d}{dz} \sum_{k=1}^{n} z^k. \tag{10}$$

After a number of transformations, from Eqs. (3) and (10) we obtain

$$M_n = P_1 R_1 \frac{nz^{n+1} - (n+1) z^n + 1}{(1-z)^2}. \tag{11}$$

With $n \to \infty$ from Eqs. (9) and (11) we obtain

$$\tilde{R}_\infty = \frac{P_1 R_1}{1-z}, \tag{12}$$

$$M_\infty = \frac{P_1 R_1}{(1-z)^2}. \tag{13}$$

C. Analysis of Derived Equations

Let us first consider the special case of $P_1 = 1$. In this case from Eqs. (5), (9), (12) and (13) we find

$$\tilde{R}_n = 1 - (1 - R_1)^n, \tag{14}$$

$$\tilde{R}_\infty = 1, \tag{15}$$

$$M_\infty = \frac{1}{R_1}, \tag{16}$$

i.e., we derive the well known equations for the case of independent shots.

Let us now consider the case of $P_1 \neq 1$, while $R_1 = 1$. In this case, from Eqs. (5), (9), (12) and (13) we obtain

$$\tilde{R}_n = P_1, \tag{17}$$

$$M_n = P_1, \tag{18}$$

i.e., the results are independent of n. This is physically evident, since when $R_1 = 1$ the target will either be damaged with the first shot or the weapon will fail on that shot and the firing operation will be curtailed.

In the subject case we have derived Eq. (18) for M_n from which it follows that $M_n \leq 1$, which is in agreement with the physical sense of the quantity M_n – the average consumption of projectiles fired from a single weapon.

We are frequently interested in the average consumption M'_n of projectiles per single damaged target. For this average consumption the following equation is obviously valid:

$$M'_n = \frac{M_n}{\tilde{R}_n}. \tag{19}$$

From Eqs. (9) and (11) we have

$$M'_n = \frac{nz^{n+1} - (n+1) z^n + 1}{(1-z)(1-z^n)}. \tag{20}$$

With $n \to \infty$, we thus find

$$M'_{\infty} = \frac{1}{1-z}. \tag{21}$$

In conclusion, let us examine a numerical example.

EXAMPLE 1. The probability $P_1 = 0.9$ of faultfree operation of the launching installation during the shot and the conditional probability $R_1 = 0.7$ of target damage with a single shot are known. Let us determine the limit values of the conditional target damage probability ($\tilde{R}_{\infty}$) and the mathematical expectation of the shots for target damage (M_{∞} and M'_{∞}) with an infinite increase in the number n of missiles available to the weapon.

Solution. From Eqs. (5), (12), (13) and (21) we find $z = 0.27$, $\tilde{R}_{\infty} = 0{,}86$, $M_{\infty} = 1{,}18$, $M'_{\infty} = 1{,}37$,

It is interesting to note that with independent shots $\tilde{R}_{\infty} = 1$. Here the quantity $\tilde{R}_{\infty} < 1$ because of the low reliability of the launching installation.

With independent shots the average number of shots for one target damage with $n \to \infty$ will be $1/R_1 = 1/0.7 = 1.43$. Here $M'_{\infty} = 1.37 < 1.43$. This is explained by the fact that it is impossible to fire a large number of shots at a single target because of the low reliability of the launching installation (the average number of shots from a single launching installation is $M_{\infty} = 1.18$).

D. Firing at Several Targets

Let us consider the case in which the gunner has at his disposal n projectiles [missiles] for the weapon. The firing is being conducted against one target until it is damaged, and fire is then transferred to the next target, etc. We are required to determine the mathematical expectation M''_n of the number of targets damaged by fire from a single weapon.

It is easy to see that this problem is equivalent to the following: there are n targets and a single shot from the given weapon is fired at each of these.

In this case the probability of damaging the first target with the first shot will be P_1R_1. The probability of proper weapon operation during the second shot will be P_1^2, while the probability of damaging the second target will be $P_1^2R_1$. Analogously, for the third target we will have $P_1^3R_1$, etc.

The sought mathematical expectation is found as the sum of the probabilities

$$M''_n = P_1R_1 + P_1^2R_1 + P_1^3R_1 + \ldots + P_1^n R_1 =$$

$$= P_1R_1 \sum_{k=1}^{n} P_1^{k-1} = P_1R_1 \frac{1 - P_1^n}{1 - P_1}. \qquad (22)$$

Hence, as $n \to \infty$, we have

$$M''_\infty = \frac{P_1R_1}{1 - P_1}. \qquad (23)$$

This equation is not suitable for the case $P_1 = 1$. In this case from Eq. (22) we have

$$M''_n = nR_1 \qquad (24)$$

and

$$M''_\infty = \infty.$$

EXAMPLE 2. Find M''_∞ for the conditions of EXAMPLE 1.

Solution. From Eq. (23) for values of $P_1 = 0.9$ and $R_1 = 0.7$ we have $M''_\infty = 6.3$. This means that with an unlimited number of shots per weapon on the average a single weapon will damage only 6.3 targets (until the weapon fails). It is assumed here that the weapon that fails is not repaired during the course of the firing operation.

§3.7. EVALUATION OF EFFECTIVENESS IN CERTAIN COMBAT SITUATIONS

A. Evaluation of Firing Effectiveness for a Single Weapon Firing a Single Shot

Let us consider the case in which the firing operation is being carried out with a single weapon which has traveled a considerable distance prior to opening fire. The system is given the command to open fire at a target whose stay time τ in the fire zone is limited. As a rule, this may involve an unplanned target (gathering of people and equipment, rocket launch installations, etc.).

This assignment can be carried out only by a complex system equipped with reconnaissance facilities. For example, an airplane, a light bomber, an artillery or rocket system with nuclear capacity and with reconnaissance facilities.

The effectiveness of such systems can be compared in terms of target damage probability. The damaging of a target with a single shot under these conditions will be a complex event consisting of a number of random events.

1. A random event involving the detection of the target by the system. This event is evaluated by the detection probability $P_{obn}(t_o)$.

2. A random event involving the fact that at the instant of time $t_k = 0$ (the instant at which the command to open fire is received) the system will be ready for operation. This event is

evaluated by the probability that at the instant of time $t_k = 0$ the system will be in an operational state K_g.

3. A random event involving the fact that the system will function faultlessly for the given time t_p which is evaluated by the probability of faultless operation $P(t_p)$ during the time t_p (preparation and launch of rocket).

4. A random event consisting of the fact that the target will not leave the fire zone (if it is a moving target) or will not leave its starting position. This event is evaluated by the probability of the target staying in the firing zone (at its starting position) $P(\tau)$.

5. A random event consisting of the fact that our system will not be destroyed during the time t_p by fire from the enemy. This event is evaluated by the probability of nondamage from enemy fire $(1 - Q)$.

6. A random event consisting of the fact that the miss distance r on launch of the rocket will be less than the reference radius r_0 of the damage zone. This event is evaluated by the probability P_1 of hitting a circle with radius r_0.

7. A random event consisting of the damage of a target on launch with a miss distance r. This event is evaluated by the conditional probability of target damage $G(r)$ under the condition that the missile explodes with a miss distance r.

The total target damage probability $\tilde{R}_1$ with a single shot, with consideration given to all of these random events, is evaluated as the product of the probabilities of all of these events:

$$\tilde{R}_1(t) = P_{\text{обн}}(t) K_{\text{г}} P(t_{\text{п}}) P(\tau)(1-Q) R_1, \qquad (1)$$

where $R_1 = P_1 G(r)$ and is calculated with Eqs. (3.1.1), (3.1.6) or (3.2.1)-(3.2.29) as a function of the type of ammunition. System reliability $P(t_p)$ was considered in Chapter 1 and in the previous section. Let us examine in greater detail the probability $P(\tau)$ of the target remaining in the firing zone, the readiness factor K_g and the detection probability.

If fire is being conducted at a launching position, the time that the target stays in the fire zone is defined by the time required to prepare for firing. The usual preparation time τ_p is characterized by the average preparation time. However, we will always have a random target stay time τ in the fire zone, since random target detection may occur at the instant that it occupies its launch position at $\tau_p = 0$ and at any other instant $\tau \leq \tau_p$. We may therefore assume an exponential law for the distribution of the target stay time τ at the launch position and calculate the probability that the target will not leave its launch position during the time τ with the equation

$$P(\tau > t_{\text{п}}) = e^{-\frac{t_{\text{п}}}{\tau_0}}, \qquad (2)$$

where $\tau \geq 0$;

τ_0 is the average target stay time in the fire zone from the instant of its detection.

At any instant of time, beginning with the instant of detection, the target may leave the launch position. Here the probability of the target staying in the fire zone is a function exclusively of τ and is independent of the position of the reckoning origin.

The probability of detecting moving targets was considered in §1.6. For nonmoving targets the detection probability $P_{\text{obn}}(t)$ is a function of the reconnaissance rate λ_r and the time t spent on reconnaissance. The reconnaissance rate λ_r will be understood to refer to the number of targets spotted per unit time:

$$P_{\text{обн}} = 1 - e^{-\lambda_{\text{р}} \cdot t}. \qquad (3)$$

The readiness factor K_g must be taken into consideration in those cases in which the period of preparation does not allow enough time to permit a technical inspection of the various system elements and to carry out repairs or substitution of system elements that have broken down. Should there be enough time to check the equipment and to repair it in the case of failure, the readiness factor will be equal to unity.

EXAMPLE 1. Evaluate the firing effectiveness of a rocket system en route with a single launch of a rocket, if it is known that the conditional probability of target damage is $R_1 = 0.9$, that the effectiveness of the return fire is $Q = 0.5$, that the time required to ready the launch is $t_p = 0.5$ hr, that the average time of faultless system operation is $T = 50$ hr, that the average repair time is $T_v = 2$ hr, that the average target stay time in the fire zone is $\tau_0 = 2$ hr, that the reconnaissance rate is $\lambda_r = 0.5$ targets per hour and that the reconnaissance time is $t = 10$ hr from the instant that the system gets under way.

Solution 1. From Eq. (3) we determine $P_{\text{obn}}(t)$ for various values of the reconnaissance time

t, hr	0	2	4	6	8	10
$P_{\text{обн}}(t)$	0	0.635	0.865	0.950	0.982	0.993

2. The readiness factor

$$K_{\text{г}} = \frac{50}{50+2} = 0.973.$$

3. The probability of faultless operation is determined fr

the equation

$$P(t_{\text{п}}) = e^{-\frac{t_{\text{п}}}{T}} = 0{,}990.$$

4. The probability of the target staying in the fire zone is found from Eq. (2) (we neglect the time spent on the processing of the reconnaissance results and target indication)

$$P(\tau > t_{\text{п}}) = e^{-\frac{t_{\text{п}}}{\tau_0}} = 0{,}775.$$

5. The effectiveness of the system firing at a launch position as a function of the reconnaissance time t is determined from Eq. (1) as $R_1(t) = 0.635 \cdot 0.973 \cdot 0.990 \cdot 0.775\ (1 - 0.5) \cdot 0.9 = 0.213$, etc.

t, hr	0	2	4	6	8	10
R_1	0	0.213	0.290	0,319	0,330	0,333

Hence we can see that the firing effectiveness of the system does not exceed 33%, whereas the conditional probability of target damage $R_1 = 0.9$. The basic factor reducing firing effectiveness in this example is the effectiveness $Q = 0.5$ of the countermeasures.

B. Evaluation of Firing Effectiveness with Several Shots

With several shots under complex combat conditions the same random factors which we considered in the previous item will affect the firing effectiveness of the system. Moreover, we will also find a relationship between the shots. However, the relationship between the shots may occur either at a high firing rate or with a low reliability of system operation (see §3.6). With high system reliability and a low firing rate we can neglect the fact of the relationship and calculate effectivenesss with the equation

$$\tilde{R}_n(t) = P_{\text{обн}}(t) K_{\text{г}} P(t) P(\tau) \{1 - [1 - P(t_0) R_1]^n\} (1 - Q), \qquad (4)$$

where $P(t)$ is the probability of the system functioning faultlessly during time t;
$P(t_0)$ is the probability of the rocket functioning faultlessly during the flight time t_0;
n is the number of rockets launched at a single target;
t is the time that the system is in operation against a single target.

The remaining parameters have the same significance as in the previous item. We can use Eq. (4) to calculate the firing effectiveness of an antiaircraft system or of an ATGM [antitank guided missile] (ПТУРС). The detection probability in this case is calculated with Eqs. (1.6.2) and (1.6.18).

Manu-script Page No.	Transliterated Symbols
170	р = r = razryv = explosion
170	рд = rd = razryv po dal'nosti = explosion, distance
170	рв = rv = razryv po vysote = explosion, altitude
185	ср = sr = sredniy = average, mean
191	н = n = normal'nyy = normal
191	г = g = gruppa = group
200	г = g = gotovnost' = readiness
203	обн = obn = obnaruzheniye = detection
203	о = o = obnaruzheniye = detection
203	к = k = komanda = command
204	п = p = podgotovka = preparation [training]
205	р = r = razvedka = reconnaissance
205	в = v = vosstanovleniye = recovery [repair]

Chapter 4

APPLICATION OF THE METHODS FROM THE THEORY OF MASS SERVICE ENGINEERING [QUEUEING] FOR THE SOLUTION OF PROBLEMS IN OPERATIONS RESEARCH

§4.0. INTRODUCTION

At the present time the ideas and methods of the theory of queueing are finding increasing acceptance throughout in practice, including in operations research. This chapter provides a brief discussion of certain results and methods from the theory of queueing which may be applied to the solution of military problems. They are all reduced to calculational relationships.

The chapter demonstrates the approaches and methods of formulating and solving certain military engineering problems associated with the theory of queueing, and these are all illustrated with appropriate examples. In the last section we present the fundamental principles for the solution of more complex problems on a computer by the method of statistical tests (the Monte Carlo method). At the end of the book and in this chapter, during the course of the discussion of the material, we provide references to available Russian literature on the problems of the theory of queueing touched upon here. The reader may refer to these citations if he desires a more extensive and more thorough familiarization with the methods and results of this theory.

§4.1. FUNDAMENTAL CONCEPTS OF THE THEORY OF QUEUEING

The theory of queueing was developed only recently. Its development was brought about initially by the urgent needs of telephone communications, and then in physics, efficient servicing of the population (stores, cashiers offices, airports, etc.), repair and servicing of machine tools, etc.

In military affairs the methods of the theory of queueing may be employed to evaluate the effectiveness of an antiaircraft defense AAD [ПВО] system for various sites on the basis of which requirements are worked out with respect to antiaircraft weapons, reliability of aerial reconnaissance, to study the firing effectiveness of antitank facilities, the transmission capacity and the quality of various forms of guidance systems, for the determination of optimum organization in the repair of combat equipment and systems for the supply to troops of ammunition and other forms of combat equipment, the forecasting of peak loads at evacuation centers, hospitals, decontamination centers, etc. [66].

As we can see from this brief listing, the range of problems of applied military nature which may be solved by means of the methods of queueing is rather extensive.

Before turning to the direct application of the results and the methods of the theory of queueing and examples of its military application, we must become familiar with its fundamental concepts and terminology.

In solving the above-enumerated applied military problems we define the term "*queueing*" to include the firing at aerial targets by an antiaircraft defense system, the observation of ground targets by aerial reconnaissance facilities, the repulsion of an armored attack by a system of antitank facilities, the processing of reconnaissance and similar data by a control system, the repair of military hardware, etc.

The queueing system includes devices (lines, flows, etc.) to provide service. If we again turn to examples from the area of military affairs, these include antiaircraft defense systems, aerial reconnaissance facilities, antitank weapons, industrial assembly lines [flows] or weapon repair shops, etc.

The function of any queueing procedure is the satisfaction of imposed requirements (requisitions). In military affairs these requirements (requisitions) for service include aerial targets in the antiaircraft defense zone, tanks in the operating zone of antitank facilities, targets for aerial reconnaissance, weapons requiring repair, etc. These requirements (requisitions) are introduced into the system to form a certain time sequence of events which will be referred to as a *flow*.

Those requirements which are imposed on the system form the input flow. However, all requirements are not serviced by the system. Some of these, rejected for certain reasons, are removed from the system without service. For example, when an enemy force of aircraft attacks a site, the antiaircraft systems are not always capable of firing at these aircraft. Some of these aircraft penetrate to the target, forming an output flow of aircraft which have not been fired upon (unserviced requirements).

The output flow may also consist of serviced requirements (aircraft and tanks against which fire has been directed, repaired weapons, etc.).

A. Classification of Queueing

All queueing systems can be divided into two major groups: *uniform* and *nonuniform*. The former consists of uniform servicing devices, while the latter consists of nonuniform devices. For example, if an antitank defense system consists of uniform antitank facilities, it will be a uniform queueing system.

The service process itself may consist of a number of sucessive phases. In this event, if there are several such phases, queueing systems are known as multiphase systems. For example, an antiaircraft defense system may be regarded as consisting of

a guidance system and a fire control system. Targets appearing in an antiaircraft defense zone are initially detected and distributed by the guidance system among the antiaircraft fire systems (the first phase) and then fired upon by these (the second phase).

The operational feature of multiphase systems involves the fact that the servicing facilities of each subsequent phase become operational only when the requirement (requisition) of the previous phase has been satisfied.

In terms of the time for which a requirement remains in effect in the servicing sphere, all systems can be divided into three major groups:

systems with failure;

systems with limited expectation time;

systems with unlimited expectation time.

Systems with failure are those in which any new input requirement for service, on finding all facilities occupied, leaves the system. An example of such a system is an antiaircraft defense system in which the stay time for a target in the firing zone is small and commensurate with the time required to carry out the firing operation. In this case, the enemy aircraft (or some other aerial attacking device), finding the antiaircraft systems engaged in firing at other aircraft, passes through the antiaircraft defense zone unharmed.

The opposite of the above system is a queueing system with an unlimited time of expectation for requirements (requisitions) in sequence (a system with expectation). The operational feature of such a system involves the fact that an input requirement, finding all service facilities occupied, must await its turn until some of the servicing units are freed. As an example of such a system we can cite a control system processing reconnaissance results and data on the position and status of friendly forces, repair workshops, etc. In the latter case, weapons coming in for repair, if flows engaged in the repair of earlier arrived equipment are occupied, must await their turn and weapons thus are accumulated in large quantities.

Finally, systems with a limited expectation time occupy an intermediate position. Requirements entering such a system, on finding all devices occupied, take their turn. However, the requirements remain in this system for a limited time, after which, unable to await service, they leave the system. The fundamental relationships derived for these systems to describe their function may be used to obtain similar relationships for the earlier considered systems. As an example of such a system we can cite the grouping of antitank facilities with rather great firing range. The time in which an enemy's tanks remain in the firing zone is rather great, but limited. For systems with failure the time of requirement expectation on an in-turn basis is equal to $t_{ozh} = 0$.

Each of the systems may vary with respect to the nature of the requirement to be serviced:

the devices are connected for servicing in rigorous sequence (for example, in numerical order). This occurs when the system consists of various types of uniform weapons with various advantageous characteristics of their combat application;

the devices begin to service new input requisitions as they become free (for example, industrial repair assembly lines);

the devices are actuated in random order (for example, antiaircraft systems in firing at targets in the case of a strong attack from the air).

In systems with expectation and limited expectation time we can determine the variations from the sequence with which the requirements are accepted for servicing:

requirements for servicing are accepted in the sequence of their input into the system (arrival of a malfunctioning weapon for repair);

preference for service is given to those requirements which have minimum time to failure (in the firing zone for antitank facilities it is advisable, first of all, to fire at those tanks which are closest to the antitank facilities and capable most rapidly of penetrating the defense perimeter);

requirements for service are accepted in random order (as an example we can cite the antiaircraft defense system for a site in repelling an enemy aerial attack).

A general feature of all problems associated with queueing is the random nature of the studied phenomena. The number of requirements for service and the magnitudes of the time intervals between these on input into the system are random. The servicing time, and in certain systems with limited expectation time, the expectation time as well, are also subject to random fluctuations, with these random fluctuations, however, not in the nature of small perturbations. Quite the opposite, this basic feature of the subject processes imposes a specific mark on the properties of the derived relationships.

B. Characteristics of Requirement Flow

In practical terms, virtually all problems pertaining to the theory of queueing have been reduced to final calculational formulas and these, having found practical application, proceed from the position that the input flow is the simplest (Poisson). The simplest flow exhibits three basic properties: steadiness, uniqueness and an absence of aftereffects. A random flow is referred to as *a steady flow* if its probability regime does not vary with time.

If we plot equal but nonintersecting time intervals τ (Fig. 4.1.1) on the time axis, the probability of the event – the ap-

pearance during these intervals of a specific number of requirements – depends for the given flow on the magnitude of τ and is independent of the position of this interval on the time axis (from the instants of time t_1, t_2, t_3, etc.).

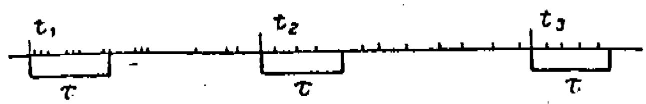

Fig. 4.1.1

For the simplest flow the probability of the appearance within a time interval of duration τ of precisely k requirements is determined with the Poisson formula

$$P_k(\tau) = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau}, \qquad (1)$$

where $\lambda > 0$ is a constant number whose significance will be clarified below.

Absence of aftereffects involves the fact that the probability of the appearance of a specific number of requirements within the time segment τ is independent of the number of requirements that have already arrived in the system earlier and is independent of the previous history of the subject phenomenon. The absence of aftereffects assumes the mutual independence of the progress of the flow in nonoverlapping time interval. *Uniqueness* of the requirement flow indicates the practical impossibility of the appearance of two or more requirements at a single instant of time. If we denote the probability of the appearance of more than one requirement during the time Δt by $P_{>1}(\Delta t)$, the uniqueness condition is written as follows:

$$\frac{P_{>1}(\Delta t)}{\Delta t} \to 0 \text{ as } \Delta t \to 0.$$

The simplest flow is thus a steady flow, one that is unique without aftereffects. The derivation of the equations for the simplest flow is demonstrated extensively in available literature on the theory of queueing and the reader may find this material in Reference [80].

An important characteristic of the flow is its rate, which is defined as the mathematical expectation of the number of requirements per unit time. For the simplest flow, the average number of requirements arriving during the time t is equal to

$$M[\mu(t)] = \sum_{k=1}^{\infty} kP_k(t) = e^{-\lambda t} \sum_{k=1}^{\infty} k \frac{(\lambda t)^k}{k!} = \lambda t, \qquad (2)$$

where $\mu(t)$ is the flow rate;
λ is the flow parameter.

The flow parameter in the theory of queueing is defined as the limit of the ratio for the probability of the appearance $\pi_1(t)$ during the time Δt of at least one requirement to the time Δt

$$\lim_{\Delta t \to 0} \frac{\pi_1(t)}{\Delta t} = \lambda. \qquad (3)$$

For the simplest flow the rate is equal to its parameter. For other steady flows we always have the inequality $\mu \neq \lambda$. Two simplest flows differ from each other only in their parameters. Thus to assume a simplest flow it is enough only to assume its parameter λ.

Graphically, the simplest flow, as well as other forms of flow, can be presented in the form of a curve of random functions having discrete whole nonnegative values (Fig. 4.1.2). The height of each step in this graph is equal to unity (the appearance of a requirement), and the length of the step defines the time interval between two successive requirements. The magnitudes of these intervals are random quantities with an exponential distribution law having the parameter λ.

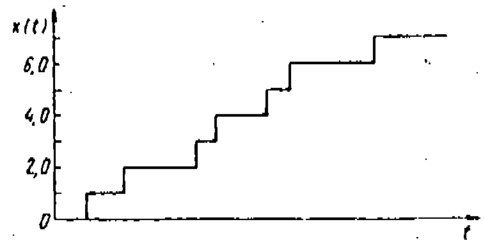

Fig. 4.1.2

An exponential distribution exhibits one interesting property — the distribution of the duration for the remaining part of the time prior to the onset of the subsequent event is independent of the amount of time that has passed since the instant of the previous event. This property enhanced extensive application of the exponential distribution in the theory of queueing. Proof of this property may be found in References [80, 66].

Particular attention is being devoted to the simplest flow because it has found overwhelming use in various applications. However, experience in the study of real flows demonstrates that these may not always be presented in the form of the simplest. Indeed, we should expect certain aftereffects, nonuniqueness and nonsteadiness in the flow, and these cannot always be neglected.

The appearance of requirement flows with limited aftereffects will be examined on the example of the antiaircraft defense of a major target. According to data from [62] the antiaircraft defense system of such a target, as a rule, consists of several echelons. Let the simplest flow of targets approach the first echelon of the antiaircraft defense system. All of the targets will not be damaged as they pass through the first echelon. The output flow of targets will now be a flow with limited aftereffects (a Palm

flow). If a Poisson flow of targets approaches the first echelon, as it passes through the antiaircraft defense elements an increasing number of voids will be formed within the flow. The farther the flow of targets passes through the echelons of the antiaircraft defense, the larger the number of voids and accumulations will form. This is the meaning of aftereffect.

Nonsteadiness of the flow with respect to time appears particularly strongly when it is considered over the course of a large period of time. As the time segment is reduced the nonsteadiness of the flow, as a rule, weakens.

Departure from uniqueness may be demonstrated on the same example of antiaircraft defense. Indeed, under practical conditions, two or more aircraft may simultaneously enter the antiaircraft defense zone.

However, the simplest flow continues to be used for a number of circumstances:

1. For other forms of flow we have not as yet derived simple formula relationships for a quantitative evaluation of the functioning quality of queueing systems.

2. It is more difficult to adapt queueing systems to the simplest flow. Therefore in designing servicing facilities in this case we figure on their operation under the most difficult of conditions. If the servicing facilities are designed for this most disadvantageous case, the servicing by the system of other random requirement flows at the same rate will be more reliable. This was the conclusion of I.N. Kovalenko [121].

3. The simplest flow in the theory of queueing plays the same role as the normal distribution function for random quantities in the theory of probabilities. In combining several random flows we form a total flow which, in terms of its characteristics, approaches the simplest flow (see [121]).

However, in actual practice there may arise a need to study the work of queueing systems which receive requirement flows substantially different from the simplest, and also other flows which have been sufficiently well studied. In this case the operation of the queueing system may be analyzed by means of the method of statistical tests (the Monte Carlo method) in which it is frequently advantageous to use digital computers (see Chapter 2).

C. Servicing Time

Servicing time is the most important characteristic of any apparatus (line) for the servicing of a system and defines its transmission capacity. Servicing time is a random quantity. This is a result of the instability of servicing facility operation (particularly with participation of a human being or a staff of people) and because of the nonidentity of the requirements coming into the system. For example, in repulsing an enemy aerial attack by the antiaircraft defense system of a target the servicing time is the time required for each antiaircraft system to fire at the

aerial targets. Naturally, the time from each firing operation at each new target by the system will vary for a variety of reasons. With respect to an antiaircraft artillery system the scattering in the time required to fire at aerial targets will be governed by changes in range and firing parameters such as velocity and target maneuvering, variations in the time required for preparation for firing, reloading time, the time required for the transfer of fire, etc.

The magnitude of the servicing time t_{obs} therefore should be assumed to be a random quantity whose total characteristic is the distribution function

$$F(t)=P[t_{обс}<t], \tag{4}$$

where $P[t_{obs} < t]$ is the probability of the event that the servicing time t_{obs} will not exceed a certain quantity t.

From physical considerations the servicing time may not be a negative quantity, i.e., when $t_{obs} \leq 0$ we have $F(t) = 0$. The servicing time distribution function is determined experimentally by statistical methods of analyzing the numerical values of the servicing time for actual systems. The distribution functions may be of the most varied kind.

However, both in theoretical applications, and particularly in practical applications, the exponential law has gained extensive acceptance. With an exponential distribution function all results are considerably simplifed, whereas development of methods for the solution of queueing problems with an arbitrary distribution function for the servicing time encounters tremendous difficulties. The exponential distribution function has the form

$$F(t)=1-e^{-\mu t}, \tag{5}$$

where $\mu = 1/\bar{t}_{obs}$ is a positive constant quantity. The quantity $\bar{t}_{obs}$ is equal to the mathematical expectation of the servicing time.

The exponential distribution function of the servicing time assumes that a significant fraction of the requirements will always be serviced rapidly, which is not always in agreement with actual practice. A.K. Erland therefore proposed the assumption of a distribution density for the servicing time with the formula

$$\begin{aligned}\varphi_k(t)&=\frac{(\mu k)^k}{\Gamma(k)}e^{-\mu k t}t^{k-1} \text{ when } t>0,\\ \varphi_k(t)&=0 \text{ when } t\leqslant 0.\end{aligned} \tag{6}$$

It can be demonstrated that $\varphi_k(t)$ represents the distribution density of the sum of k independent random quantities with an exponential distribution function. The form of the function $\varphi_k(t)$ is shown in Fig. 4.1.3. This distribution of the servicing

time is closer to the actual distribution. Queueing systems with an exponential distribution function exhibit an important property which must be borne in mind in evaluating armament effectiveness.

Let a requirement enter a queueing system consisting of n various units. The servicing time for the requirement by each of the devices is subject to an exponential law with the parameter μ. The servicing is concluded as soon as one of the devices has completed its servicing task. It can be demonstrated (see [121]) that for this case the servicing function by each of the devices will also be exponential

$$F(t_{\text{обс}} < t) = 1 - e^{-(\mu_1+\mu_2+\ldots+\mu_n)t} \tag{7}$$

with the parameter

$$\mu = \sum_{i=1}^{n} \mu_i. \tag{8}$$

If all of the devices have identical productivity, $\mu = n\mu_i$. This means that with the simultaneous servicing of the requirement by several devices the average servicing time diminishes by a factor of n in comparison with the servicing time of a single device. It should be noted that the dispersion in this case diminishes by a factor of n^2. This property can be illustrated by examples from military activities.

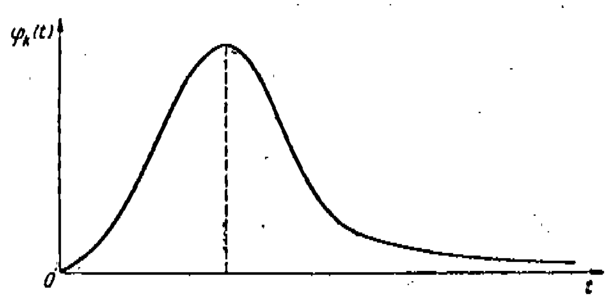

Fig. 4.1.3

Such situations arise in firing at a single aircraft by several antiaircraft systems, in the simultaneous bombing of a vessel or a similar target by several bombers, in firing at a tank by several antitank facilities, etc. [121]. In all of these cases the servicing of the requirement (firing, bombing) is carried out until the attack target is damaged. This is seen in the extensive use in combat of massed combined strikes at an enemy.

§4.2. EVALUATION OF THE EFFECTIVENESS OF ANTIAIRCRAFT DEFENSE WHERE THE TARGET REMAINS IN THE FIRING ZONE FOR A LIMITED PERIOD OF TIME

Let us consider the antiaircraft defense system of an objective, consisting of antiaircraft weapons for which the target stay time in the firing zone is commensurate with the time required for reliable objective damage [66, 17]. This situation may be governed by

a combination of target velocity, flight and altitude parameters with the tactical-technical characteristics of the antiaircraft weapon.

Formulation of the problem. The antiaircraft system consists of n uniform antiaircraft units, each of which can simultaneously fire only at a single target. The enemy has attacked the antiaircraft and the site which it is protecting at a rate λ. We are required to solve the following important problems:

evaluate the anticipated number of aerial targets downed by the antiaircraft system in repulsing the attack;

determine the anticipated quantity of aerial targets capable of penetrating the antiaircraft system to the site and determine the combat assignment;

evaluate the basic tactical-technical characteristics of the antiaircraft weapon from the standpoint of the possible improvement in the effectiveness of the antiaircraft system;

evaluate the combat potentials of the attacking facilities with respect to penetrating the antiaircraft defense system;

determine the necessary composition of the antiaircraft defense facilities with the required tactical-technical characteristics, proceeding from the standpoint of the required effectiveness.

We assume that the flow of targets in the attack is of the simplest kind. Adoption of this assumption is based on the following points:

despite the need to maintain distance and intervals in the attack, their magnitudes exhibit random deviations from those required;

under the action of the fire from the antiaircraft defense facilities, the combat formations of the attacking aircraft are disrupted;

if the enemy attacks the protected site from several directions, the total flow of attacking aircraft is close to a Poisson flow.

As mentioned earlier, the simplest flow of attacking targets is the most difficult from the standpoint of repulsing the attack by an antiaircraft defense system. This makes it possible to evaluate the effectiveness of the antiaircraft defense for a site under more difficult conditions. In solving this problem we give no consideration to the return fire of the enemy.

Each target appearing in the antiaircraft defense zone is immediately fired upon by one of the systems. If all of the systems are already firing at targets, aircraft newly appeared in the firing zone will penetrate to the protected objective.

We assume that the time required to fire at an aircraft by an antiaircraft system is a random quantity and is subject to the exponential distribution function having the parameter ν. Therefore, the probability that the time required to fire at the target will not exceed t is determined from the expression

$$P(t) = 1 - e^{-\nu t}.$$

The probability of the opposite event is equal to $g(t) = e^{-\nu t}$. The antiaircraft defense system may be found in the following situations:

A_0 denotes all systems not engaged in firing operation;
A_k – k systems are firing, with the remaining open $k = 1, 2, 3, \ldots, (n-1)$;
A_n denotes all systems firing.

Let us derive the differential equations of these conditions for an antiaircraft defense system.

Let Δt denote a very small time interval. We derive the equation of state for A_0. It is possible in the following nonsimultaneous cases:

at the instant of time t all systems are not firing. During the time Δt not a single enemy aircraft has appeared in the antiaircraft defense zone. The probability of this event is equal to

$$P_0(t)\, e^{-\lambda \Delta t}; \tag{1}$$

at the instant of time t one of the systems is engaged in firing. During the time Δt no new targets appeared in the antiaircraft defense zone, and the system has concluded firing operation against the target. The probability of this event is equal to

$$P_1(t)\,(1 - e^{-\nu \Delta t})\, e^{-\lambda \Delta t}, \tag{2}$$

where $P_1(t)$ is the probability of the case that a single system is firing.

Since the subject events are nonsimultaneous, we can derive the equation of state for A_0

$$P_0(t + \Delta t) = P_0(t)\, e^{-\lambda \Delta t} + P_1(t)\,(1 - e^{-\nu \Delta t})\, e^{-\lambda \Delta t}, \tag{3}$$

where $P_0(t+\Delta t)$ is the probability of not a single one of the systems firing during the time $(t + \Delta t)$;
$P_0(t)$ is the probability of finding the antiaircraft defense system in state A_0;
$e^{-\lambda \Delta t}$ is the probability of not a single target appearing in the firing zone during Δt;
$1 - e^{-\nu \Delta t}$ is the probability that one of the systems will conclude its firing operations against a target during the time Δt.

The quantity $e^{-\lambda \Delta t}$ can be presented in the form of a series

$$e^{-\lambda\Delta t} \approx 1 - \lambda\Delta t + \ldots,$$

while

$$1 - e^{-\nu\Delta t} \approx \nu\Delta t + \ldots.$$

Considering the smallness of the quantity Δt, we can present Eq. (3) in the form

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\,\nu\Delta t\,(1-\lambda\Delta t) \tag{4}$$

We divide both parts of the equation by Δt and, passing to the limit, we obtain

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \lambda P_0(t) + \nu P_0(t);$$

as $\Delta t \to 0$ we obtain

$$P'_0(t) = -\lambda P_0(t) + \nu P_1(t). \tag{5}$$

Let us examine state A_k. It is possible in three nonsimultaneous cases:

at the instant t we have k systems engaged in firing, and not a single enemy aircraft entered the firing zone of the antiaircraft defense system during the time Δt nor did any of the systems conclude firing operations:

$$P_k(t)(1-\lambda\Delta t)(1-k\nu\Delta t);$$

at the instant t the antiaircraft defense system was in state A_{k-1}. One more target entered the firing zone during time Δt, but none of the systems concluded firing operations against its target:

$$P_{k-1}(t)\,\lambda\Delta t\,(1-k\nu\Delta t);$$

at the instant t the system was in state A_{k+1}. One of the systems became free during time Δt, and no new targets appeared in the firing zone:

$$P_{k+1}(t)(1-\lambda\Delta t)(k+1)\,\nu\Delta t.$$

Then

$$\begin{aligned} P_k(t+\Delta t) = {} & P_k(t) \times (1-\lambda\Delta t)(1-k\nu\Delta t) + \\ & + P_{k-1}(t)(1-k\nu\Delta t)\,\lambda\Delta t + P_{k+1}(t)(1-\lambda\Delta t)(k+1)\,\nu\Delta t. \end{aligned} \tag{6}$$

After analogous transformations we obtain

$$P'_k(t) = -(\lambda + k\nu)P_k(t) + \lambda P_{k-1}(t) + P_{k+1}(t)(k+1)\,\nu. \tag{7}$$

This equation is valid for the case

$$0 \leqslant k \leqslant n.$$

Let us examine the extreme state A_n. It is possible in the following nonsimultaneous cases:

at the instant of time t the antiaircraft defense system was in state A_n. Not a single weapons unit became free during the time Δt

$$P_n(t)(1-n\nu\Delta t);$$

at the instant of time t the system was in state A_{k-1}. One more aircraft appeared in the antiaircraft defense zone during the time Δt and none of the weapons unit became free

$$P_{n-1}(t)(1-n\nu\Delta t)\,\lambda\Delta t.$$

After appropriate transformations and passing to the limit as $\Delta t \to 0$, we obtain

$$P'_n = -n\nu P_n(t) + \lambda P_{n-1}(t). \tag{8}$$

All of these equations together have come to be known as the *Erlang* system of equations.

Determination of the Steady Solution

The steady solution is understood to refer to that solution which corresponds to a formulated and steady process in the absence of any kind of transient phenomena characteristic of the start of servicing.

In determining the steady solution we proceed from the steady-state process, i.e., the state of the system as $t \to \infty$.

Let us examine the functioning of an antiaircraft defense system in the repulsion of a prolonged attack. In this case

$$P_k(t) \to P_k = \text{const},\ P'_k(t) \to 0,\ k = 0,\ 1,\ 2,\ \ldots,\ n.$$

The system of differential equations

$$\begin{gathered} P'_0(t) = -\lambda P_0(t) + \nu P_1(t), \\ P'_k(t) = -(\lambda + k\nu)P_k(t) + \lambda P_{k-1}(t) + P_{k+1}(t)(k+1)\nu \\ \cdots\cdots\cdots\cdots\cdots \\ P'_n(t) = -n\nu P_n(t) + \lambda P_{n-1}(t) \end{gathered} \tag{9}$$

is then transformed into the system of algebraic equations

$$\begin{gathered} -\lambda P_0 + \nu P_1 = 0, \\ \cdots\cdots\cdots\cdots \\ -(\lambda + k\nu)P_k + \lambda P_{k-1} + (k+1)\nu P_{k+1} = 0, \\ \cdots\cdots\cdots\cdots \\ -n\nu P_n + \lambda P_{n-1} = 0. \end{gathered} \tag{10}$$

As demonstrated in [121, 80], from this system we can determine the probabilities of the various states.

The probability of state P_k (k units are firing) is determined from the formula

$$P_k = \frac{\frac{\alpha^k}{k!}}{\sum_{k=0}^{n} \frac{\alpha^k}{k!}}, \tag{11}$$

where α is the average number of targets entering the antiaircraft defense zone during the average time required for the weapons unit to fire at a single target

$$\alpha = \frac{\lambda}{\nu};$$

λ is the average number of targets entering the antiaircraft defense zone per unit time

$$\nu = \frac{1}{\bar{t}_{\text{обс}}};$$

$\bar{t}_{obs}$ is the average time required by the weapons unit to fire at the target.

The characteristic of the antiaircraft defense system represented by the probability of all weapons units being simultaneously engaged in firing at targets may be of interest. This probability may be referred to as the probability of the antiaircraft defense system permitting the passage of unharmed targets

$$P_{\text{проп}} = \frac{\frac{\alpha^n}{n!}}{\sum_{k=0}^{n} \frac{\alpha^k}{k!}}. \tag{12}$$

Formula (12) has been derived in the assumption that the servicing time is subject to an exponential distribution function.

B.A. Sevost'yanov [12] proved a more general result according to which the Erlang formula remains valid for an arbitrary servicing time distribution function (see [121]). Tables (11.5) have been compiled for Formulas (11) and (12). The probability that each target will have been fired upon is equal to

$$P_{\text{обс}} = 1 - P_n, \tag{13}$$

and the probability that the target will be downed is determined from the formula

$$P_{\text{сб}} = P_{\text{обс}} P, \tag{14}$$

where P is the probability of the weapons unit damaging each target as it fires at that target.

The mathematical expectation of the number of downed targets during the attack is equal to

$$M_{сб} = NP_{сб}. \tag{15}$$

The mathematical expectation of the number of aircraft permitted to pass through the covered target without harm is equal to

$$M_{проп} = N(1 - P_{сб}). \tag{16}$$

The mathematical expectation of the number of weapons unit engaged in firing is equal to

$$M_k = \sum_{k=1}^{n} kP_k = P_0 \sum_{k=1}^{n} \frac{\alpha^k}{(k-1)!}. \tag{17}$$

The average load of each weapons unit during an attack is

$$\Delta T\%_0 = \frac{M_k}{n}.$$

Earlier we presented Formulas (15) and (16) which were derived for the steady solution, i.e., for an attack of great duration. Formulas (15) and (16) are therefore approximate. These relationships may be used for practical purposes, if the attack time t_{nal} is greater by a factor of 2-3 than the average time required by the unit to fire at a single target.

From the results of the calculations carried out by the method of statistical tests we have that when $t_{nal} > 2\bar{t}_{obs}$ the nonsteadiness of the subject process has no significant effect on the results and, depending on the relationship between the quantities λ and ν, the calculational errors M_{sb} and M_{prop} do not exceed 5-10%.

The utilization of the derived relationships will be examined through examples.

EXAMPLE. A site protected by antiaircraft defense is attacked by an enemy from the air at an average rate of $\lambda = 4$ aircraft/min. In the attack area the site is protected by 6 units ($n = 6$) with an average time for firing at a single target given as $\bar{t}_{obs} = 0.5$ min. The attack is carried out by 24 aircraft ($N = 24$). We are required to evaluate the effectiveness of the antiaircraft protection system for the site when $P = 0.7$.

In solving the problem we proceed from the assumption that the aircraft attacking the site represent a Poisson flow. We find the auxiliary parameter

$$\alpha = \lambda t_{обс} = 4 \cdot 0{,}5 = 2 \text{ aircraft}.$$

To determine the probabilities of various states for the antiaircraft defense system we use the formula

$$\frac{P_k}{P_0} = \frac{\alpha^k}{k!},$$

while the results of the calculations are presented in Table 4.2.1.

TABLE 4.2.1

Number of units engaged in firing	$\frac{P_k}{P_0}$	P_k	kP_k
0	1	0,136	0
1	2	0,272	0,272
2	3	0,272	0,544
3	1,333	0,181	0,543
4	0,666	0,091	0,363
5	0,267	0,036	0,180
6	0,088	0,012	0,072
Total	7,353	1	1,975

Using the results shown in Table 4.2.1, we obtain the following characteristics for the system providing antiaircraft protection to the site in the attack area:

the probability of permitting the unharmed passage of targets to the site

$$P_n = P_{k=6} = 0{,}012;$$

the probability of directing fire at each target participating in the raid

$$P_{\text{cob}} = 1 - P_n \approx 0{,}99;$$

the probability of downing a target participating in the raid

$$P_{\text{cб}} = P_{\text{oбo}} \cdot P = 0{,}99 \cdot 0{,}7 = 0{,}7;$$

the mathematical expectation of the number of targets participating in the raid that are fired upon

$$M_{\text{oбc}} = NP_{\text{oбc}} = 24 \cdot 0{,}99 = 24 \text{ aircraft;}$$

the mathematical expectation of the number of targets participating in the raid that are downed

$$M_{\text{cб}} = P_{\text{cб}} \cdot N = 0{,}7 \cdot 24 = 16{,}8 \text{ aircraft;}$$

the mathematical expectation of the number of aircraft penetrating to the objective

$$M_{\text{npou}} = N(1 - P_{\text{cб}}) = 24(1 - 0{,}7) = 7{,}2 \text{ aircraft;}$$

the mathematical expectation of the number of weapons units engaged in firing during the course of the raid

$$M_k = 1{,}98 \text{ units.}$$

This means that during the attack each weapons unit

$$\Delta T\% = \frac{M_{\text{к}}}{n} = \frac{1,98}{6} = 33\% \text{ of the time.}$$

will be engaged in firing. The remaining time may be used to reload or for other purposes

$$T_{\text{н}} = \frac{N}{\lambda} = 6 \text{ min},$$

each of the weapons units will fire at

$$n_{\text{обс}} = \frac{N \cdot P_{\text{обс}}}{n} = \frac{24 \cdot 0,99}{6} = 4 \text{ targets},$$

spending on the average t_{obs} = 5 min on each of these targets. The remaining time of approximately 4 min may be used to reload or to fire at other targets.

Let us consider the effect of the number of weapons units in the antiaircraft defense system on the nature of its operation. Table 4.2.2 shows for purposes of comparison the probabilities of each target being fired upon and the mathematical expectation of the number of weapons units engaged in firing during the course of the raid.

It follows from the tabular data that a reduction in the number of weapons units from six to five has an insignificant effect on the effectiveness of the antiaircraft defense system. With a further reduction in the number of weapons units the effectiveness of the antiaircraft defense system diminishes more markedly. At first glance it may seem to be an unexpectedly small number of weapons units that are engaged in firing at targets during the raid. This is explained by the fact that new targets may appear in the antiaircraft defense zone while these weapons are firing and these new targets will thus pass through that zone without harm. On the other hand, because of the nonuniform load of the units and the nonuniform appearance of targets in the firing zone some units may stand idle. However, as follows from Table 4.2.2, with a reduction in the number of weapons units, the percentage of their firing load during the course of the attack increases.

TABLE 4.2.2

n	$P_{\text{обс}}$	$M_{\text{к}}$	$\Delta T\ \%$
2	0.60	1.2	60
3	0.78	1.61	54
4	0.90	1.81	45
5	0.96	1.93	39
6	0.99	1.98	33

TABLE 4.2.3

Number of units concentrated on a single target	Mathematical expectation of the number of downed aircraft (M_{ob})	
	when $P=0,1$	when $P=0,9$
$m=1$	1,2	10,9
$m=2$	2,2	11,2
$m=3$	3,0	10.5

Within the framework of the example, let us analyze the need and feasibility of concentrating the fire of several units at a single target. Let the fire of two or three firing units be concentrated at each aircraft. The average time of fire at an aircraft with this procedure is reduced in comparison with the time for a single unit by a factor of two or three (refer to the property of the exponential servicing function in §4.1), i.e., $t_{2obs} = 0.25$ min, $\bar{t}_{3obs} = 0.167$ min. The auxiliary parameter will then be equal to $\alpha_2 = 1$ aircraft, $\alpha_3 = 0.67$ aircraft. The remaining conditions of the example remain unchanged. For these conditions Table 4.2.3 shows the mathematical expectations of the number M_{sb} of downed aircraft, when the probabilities of downing an aircraft with the firing of a single unit are equal to $P = 0.1$ and 0.9.

Analysis of the results presented in Table 4.2.3 shows that with small values for the probabilities of downing a target with the fire of a single unit any concentration of fire markedly increases the effectiveness of the antiaircraft defense system. With high values for the same probabilities, this will either yield no perceptible increase in the number of downed targets in the raid (with $m = 2$), or it becomes unfeasible (with $m = 3$).

§4.3. EFFECTIVENESS OF AN ANTIAIRCRAFT DEFENSE SYSTEM SET UP IN ECHELONS CONSISTING OF SIMILAR WEAPONS UNITS

The antiaircraft defense system for a major site may be set up in the form of successive zones or echelons (Fig. 4.3.1). This is referred to as the echelon defense system. In this case the enemy aircraft, before reaching the attack objective, are compelled successively to overcome all of the echelons of the antiaircraft defense. There may be various numbers of antiaircraft facilities in each antiaircraft defense zone.

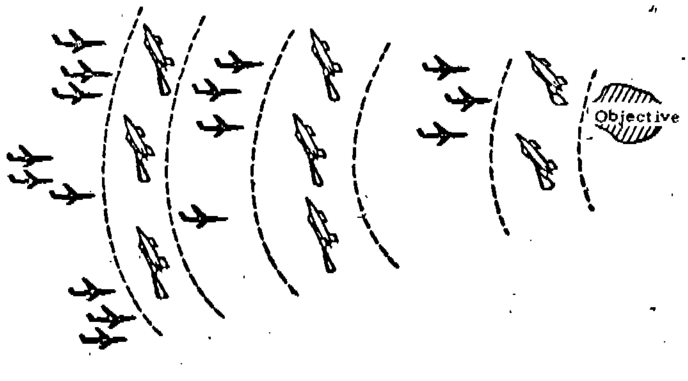

Fig. 4.3.1

Let us evaluate the effectiveness of such an antiaircraft defense system, consisting of uniform antiaircraft facilities. As was done earlier, we will assume that the enemy aircraft, carrying out the attack, form a simple flow with the parameter λ. Let us consider the case in which the time required by a unit to

fire at the target is a random quantity with an exponential distribution function having the parameter ν. Each of the units can fire simultaneously only at a single target.

In the solution of the problem we do not take into consideration the return fire of the enemy. For the enemy aircraft to overcome the first defense area (the first echelon of the antiaircraft protection) all of the antiaircraft facilities must be engaged in firing. The probability of this event for the case in which the target stay time in the damage zone of the unit is small (system with failures) is determined by means of Formula (4.2.12)

$$P_1 = \frac{\frac{\alpha^n}{n_1!}}{\sum_{k=0}^{n_1} \frac{\alpha^k}{k!}}.$$

For the aircraft to pass through the second echelon the antiaircraft facilities of both echelons must be engaged in firing. The probability of this event is equal to

$$P_2 = \frac{\frac{\alpha^{n_1}\alpha^{n_2}}{(n_1+n_2)!}}{\sum_{k=0}^{n_1+n_2} \frac{\alpha^k}{k!}},$$

where n_1 is the number of antiaircraft facilities in the first echelon;
n_2 is the same number for the second echelon.

If there are i such echelons, the probability of a target passing through to the defended site is equal to

$$P_i = \frac{\frac{\alpha^{n_1}\alpha^{n_2}\ldots\alpha^{n_i}}{(n_1+n_2+\ldots+n_i)!}}{\sum_{k=0}^{n_1+n_2+\ldots n_i} \frac{\alpha^k}{k!}}.$$

EXAMPLE. An aerial attack is being carried out against a site with an echelon antiaircraft defense system consisting of three defense zones. There are two antiaircraft weapons units in the first antiaircraft defense zone, there are three such units in the second zone and one unit in the third zone. All of the weapons units are similar. The time required to fire at a target by each of the antiaircraft facilities is random and has an exponential distribution law and the parameter $\nu = 1$ aircraft/min. The rate of the attacking aerial targets is $\lambda = 2$ aircraft-min. The probability of downing a target with firing from a unit is close to unity $P \approx 1$. Evaluate the effectiveness of the antiaircraft defense system for the objective and for each of its echelons.

The probability of the enemy aircraft penetrating through the first zone is equal to

$$P_{n_1} = \frac{\frac{2^2}{2!}}{1+2+\frac{2^2}{2!}} = 0.4.$$

This indicates that 60% of the enemy aircraft will be destroyed. The probability of the passage of the targets through the second zone is equal to

$$P_{n_2} = \frac{\frac{2^5}{5!}}{\sum_{k=0}^{k=5} \frac{2^k}{k!}} = 0.037,$$

and the probability of penetration through the entire antiaircraft defense system is

$$P_{n_3} = \frac{\frac{2^6}{6!}}{\sum_{k=0}^{k=6} \frac{2^k}{k!}} \cong 0.001.$$

This result indicates that of a thousand attacking aircraft, on the average only a single aircraft will reach its objective.

§4.4. ECHELON ANTIAIRCRAFT DEFENSE SYSTEM CONSISTING OF UNITS OF DIFFERENT TYPES

We will consider the evaluation of the effectiveness for such an antiaircraft defense system through the example of an objective whose antiaircraft protection consists of two echelons [62]. The mathematical apparatus developed for this purpose permits optimum distribution of the antiaircraft facilities into echelons, depending on the characteristics.

Let an attack be carried out against the objective in a rather narrow band in such a manner that the attacking aircraft may be fired upon as they move toward the target by one of the units in each echelon. We are required to evaluate the effectiveness of the antiaircraft defense system of the objective and to select an efficient method of distributing the antiaircraft facilities among the echelons.

Let us analyze the antiaircraft facilities with damage zones in which the target stay times are barely adequate for reliable damage. The time during which a target is fired upon by each of the antiaircraft facilities is a random quantity is subject to the exponential function having the parameters μ_1 and μ_2, respectively, for the first and second types of antiaircraft weapons. The enemy aircraft carry out the attack at a rate λ and form the simplest flow (see §4.4.1). We denote the probabilities of the state for the antiaircraft defense system in repelling the attack as follows:

P_{00} is the probability that neither the first nor the second

units are firing;

P_{10} is the probability that the first unit is firing and that the second unit is free;

P_{01} is the probability that the first unit is free and that the second unit is firing;

P_{11} is the probability that both units are firing.

A target entering an antiaircraft defense zone is first fired upon by the first unit. If that unit is already engaged in firing, any new target will fly deeper and enter the firing zone of the second unit. If the target is fired upon by the first unit and is not damaged, the second unit will no longer be able to fire on that target. The [new] target will now be fired upon by the second unit. If that unit is already engaged in firing at the previous target, the new target will pass unharmed through the antiaircraft defense zone.

We denote the states of the system as A_{00}, A_{10}, A_{01} and A_{11}. To determine the probabilities of the states we compile a system of differential equations.

State A_{00} is possible in the following nonsimultaneous cases:

during the time t the system was in state A_{00}. During the time interval Δt not a single target appeared in the antiaircraft defense system

$$P_{00}(t)(1-\lambda\Delta t);$$

during the time t the antiaircraft defense system was in state A_{10}. During the time Δt an aerial target was fired upon by the first unit

$$P_{10}(t)\mu_1\Delta t;$$

during time t the antiaircraft defense system was in state A_{01}. During the time Δt the second unit completed firing at a target

$$P_{01}(t)\mu_2\Delta t.$$

The differential equation for state A_{00} is then written in the following form:

$$P_{00}(t+\Delta t)=P_{00}(t)(1-\lambda\Delta t)+P_{10}(t)\mu_1\Delta t+P_{01}(t)\mu_2\Delta t.$$

After transformation and passing to the limit $\Delta t \to 0$ we obtain

$$P'_{00}(t)=-P_{00}(t)\lambda+P_{10}(t)\mu_1+P_{01}(t)\mu_2. \quad (1)$$

We examine state A_{01}. It is possible in the following nonsimultaneous cases:

the antiaircraft defense system during time t is in state

A_{01}. During time Δt no new targets entered the firing zone and the second unit did not complete its firing

$$P_{01}(t)(1-\lambda\Delta t)(1-\mu_2\Delta t);$$

at the instant of time t both units were firing at targets. During time Δt the first unit completed firing at its target

$$P_{11}(t)\mu_1\Delta t.$$

Hence the equation of state

$$P'_{01}(t)=-P_{01}(t)(\lambda+\mu_2)+P_{11}(t)\mu_1. \tag{2}$$

In deriving the differential equation of state A_{10} we must proceed from the fact that this state for the antiaircraft defense system is possible under the following nonsimultaneous conditions:

at the instant of time t the system was in state A_{10}. During time Δt no new targets appeared and the first unit did not complete its firing operation

$$P_{10}(t)(1-\lambda\Delta t)(1-\mu_1\Delta t);$$

at the instant of time t there were no targets in the firing zone. During the time Δt a target appeared over the first unit and it begin to fire at that target

$$P_{00}(t)\lambda\Delta t;$$

at the instant of time t both units were firing. During the time Δt the second unit completed its firing operation

$$P_{11}(t)\mu_2\Delta t.$$

Hence the equation of state

$$P'_{10}(t)=\lambda P_{00}(t)-P_{10}(t)(\lambda+\mu_1)+P_{11}(t)\mu_2. \tag{3}$$

Finally, the last state of the antiaircraft defense system is possible in the following nonsimultaneous cases:

at time t the system was in state A_{01} or A_{10}. During the time Δt new targets appeared

$$[P_{01}(t)+P_{10}(t)]\lambda\Delta t;$$

at the instant of time t both units were already conducting fire. During the time Δt neither of the units had concluded its firing operation against the targets

$$(1-\mu_1\Delta t)(1-\mu_2\Delta t)P_{11}(t).$$

Hence

$$P'_{11}(t)=\lambda[P_{01}(t)+P_{10}(t)]-(\mu_1+\mu_2)P_{11}. \tag{4}$$

The general system of equations describing the various possible states of the antiaircraft defense system is presented in the following form:

$$\begin{aligned}
P'_{00}(t) &= -P_{00}(t)\lambda + \mu_1 P_{10}(t) + \mu_2 P_{01}(t),\\
P'_{01}(t) &= -P_{01}(t)(\lambda+\mu_2) + P_{11}(t)\mu_1,\\
P'_{10}(t) &= -P_{10}(t)(\lambda+\mu_1) + P_{00}(t)\lambda + P_{11}(t)\mu_2,\\
P'_{11}(t) &= \lambda P_{01}(t) + \lambda P_{10}(t) - (\mu_1+\mu_2)P_{11}(t).
\end{aligned} \qquad (5)$$

For a steady solution the following conditions are satisfied:

$$t \to \infty,\ P'_{ij}(t) \to 0,\ P_{ij}(t) \to P_{ij} = \text{const}$$

and the system of differential equations is transformed into an algebraic system

$$\begin{aligned}
P_{00}\lambda &= \mu_1 P_{10} + \mu_2 P_{01},\\
P_{01}(\lambda+\mu_2) &= P_{11}\mu_1,\\
P_{10}(\lambda+\mu_1) &= \lambda P_{00} + P_{11}\mu_2,\\
(\mu_1+\mu_2)P_{11} &= \lambda P_{01} + \lambda P_{10}.
\end{aligned} \qquad (6)$$

In the solution of this system we determine:

the probability of an aerial target passing through the antiaircraft defense system without harm

$$P_{11} = \frac{\lambda^2}{\lambda^2 + \lambda(\mu_1+\mu_2) + \frac{\mu_1\mu_2}{\lambda+\mu_2}(2\lambda+\mu_1+\mu_2)}; \qquad (7)$$

the probability that all units are not firing

$$P_{00} = \frac{\lambda_1}{\lambda+\mu_2}P_{11}. \qquad (8)$$

From the system of equations (6) it is easy to derive the probabilities for the remaining states of the system.

EXAMPLE. Let us analyze the effectiveness of the antiaircraft defense system shown in Fig. 4.4.1.

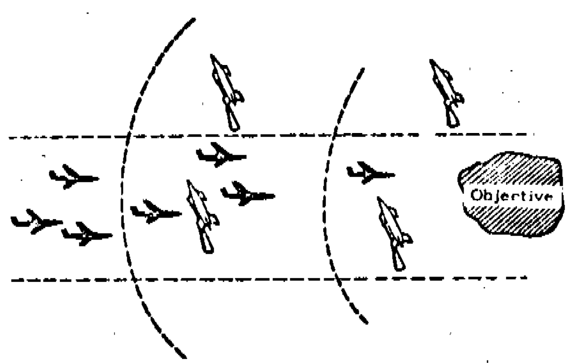

Fig. 4.4.1

Let $\mu_1 = 2$ aircraft/min, $\mu_2 = 4$ aircraft/min and $\lambda = 2$ aircraft/min. The probability of a target penetrating to the objective unharmed is equal to

$$P_{11} = \frac{2^2}{2^2 + 2(2+4) + \frac{2\cdot 4}{2+4}(2\cdot 2 + 2 + 4)} = 0{,}137.$$

Let us distribute the antiaircraft facilities differently. A unit [system] with the characteristic $\mu = 4$ aircraft/min is positioned in the first echelon, while one with the characteristic $\mu = 2$ aircraft/min is positioned in the second. The probability of an enemy aerial target penetrating to the objective being defended by the antiaircraft defense system will be different:

$$P_{11} = \frac{2^2}{2^2 + 2(4+2) + \frac{4\cdot 2}{2+2}(2\cdot 2 + 4 + 2)} = 0{,}111.$$

As we can see from the example, for proper positioning of antiaircraft facilities by echelons it is possible to raise the effectiveness of the antiaircraft defense system.

§4.5. EVALUATION OF THE EFFECTIVENESS OF THE GUIDANCE [CONTROL] SYSTEM

Let us examine a system consisting of reconnaissance facilities and fire control facilities for "ground-to-ground" units converting the reconnaissance information. The reconnaissance system, possessing certain technical facilities, detects fire facilities, command centers, troop concentrations, etc., in the enemy positions, and we will refer to these simply as targets [130].

Let the reconnaissance possess all manner of facilities which permit it to detect μ_1 targets per unit time. It is natural to assume that the time intervals between the instants of target detection are random quantities. With time the detected targets form a flow which is very close to the simplest flow. The reconnaissance data on the detected target enter the system processing reconnaissance data and the fire control system (we refer to this simply as the control system) which exhibits a limited transmission capacity for the processing of the input information per unit time. We denote the transmission capacity of the control system by μ_2. The time for the processing of the reconnaissance data for each target is a random quantity. The target data processed in the control system are distributed among the "ground-to-ground" firing units for target destruction.

Let us consider the case in which the target stay time at a single point is extremely limited and commensurate with the time needed for its detection, the processing of the initial data and the opening of fire at that target. This complex system, in first approximation, may therefore be regarded as a system with failures:

We denote the probabilities of system state as:

P_{00} denotes that the reconnaissance and control systems are free;

P_{10} denotes that the reconnaissance system is occupied with the input of information about one of the detected targets, while the control system is free;

P_{01} denotes that the reconnaissance system is free while the control system is engaged in processing target information;

P_{11} denotes that both systems are occupied.

Let us derive the differential equations of state for the control system, and these may be denoted, respectively, A_{00}, A_{01}, A_{10}, A_{11}.

System state A_{00} is possible in the following nonsimultaneous cases:

the reconnaissance and control systems are free at time t. During the interval Δt not a single target has been detected

$$P_{00}(t)(1-\lambda\Delta t);$$

the entire system at time t was in state A_{01}. During the time Δt data on the target were transmitted to the "ground-to-ground" unit to open fire

$$P_{01}(t)\mu_2\Delta t.$$

The general equation of state

$$P_{00}(t+\Delta t)=P_{00}(t)(1-\lambda\Delta t)+P_{01}(t)\mu_2\Delta t.$$

After passing to the limit as $\Delta t \to 0$ we obtain

$$P'_{00}(t)=-P_{00}(t)\lambda+P_{01}(t)\mu_2. \quad (1)$$

We have the state A_{01} for the system. It is possible in the following nonsimultaneous cases:

the system is in state A_{01}. During the time Δt no new targets have been detected and the control system is processing no data on any target

$$P_{01}(t)(1-\lambda\Delta t)(1-\mu_2\Delta t);$$

the system is in state A_{10}. During the time Δt the reconnaissance system detected and transmitted data on a target to the control system

$$P_{10}(t)\mu_1\Delta t;$$

at time t the system was in state A_{11}. During the time Δt the reconnaissance system detected and transmitted data on a target to the control system, but the latter did not accept this information since it was engaged in the processing of data on a

previous target. The input data were therefore irretrievably lost because of the brief duration of the time that the target spends in a single position

$$P_{11}(t)\mu_1\Delta t.$$

Hence the equation of state is written after the appropriate transformation

$$P'_{01}(t)=-(\lambda+\mu_2)P_{01}(t)+\mu_1P_{11}(t)+\mu_1P_{10}(t). \qquad (2)$$

Let us consider the system state A_{10}. It is possible in the following cases:

at time t the system was in state A_{00}. During time Δt a target is detected

$$P_{00}(t)\lambda\Delta t;$$

at time t the system is in state A_{10}. During the time Δt the reconnaissance system was processing no data on the target and none were transmitted to the control system

$$P_{10}(t)(1-\mu_1\Delta t);$$

at time t the system was in state A_{11}. During time Δt the control system transmitted data for firing at the target

$$P_{11}(t)\mu_2\Delta t.$$

The equation of state

$$P'_{10}(t)=\lambda P_{00}(t)-\mu_1P_{10}(t)+\mu_2P_{11}(t). \qquad (3)$$

And the last state is A_{11}. It is possible in the following nonsimultaneous cases:

the system is in state A_{01}. During time Δt new target data are received

$$P_{01}(t)\lambda\Delta t;$$

at time t the system was in state A_{11}. During time Δt no target data were processed by the reconnaissance and control systems

$$P_{11}(t)[1-(\mu_1+\mu_2)\Delta t].$$

Then

$$P'_{11}(t)=\lambda P_{01}(t)-(\mu_1+\mu_2)P_{11}(t). \qquad (4)$$

The various possible system states are described by all of the differential equations

$$
\begin{aligned}
P'_{00}(t) &= -\lambda P_{00}(t) + \mu_2 P_{01}(t), \\
P'_{01}(t) &= -(\lambda + \mu_2) P_{01}(t) + \mu_1 P_{11}(t) + \mu_1 P_{10}(t), \\
P'_{10}(t) &= \lambda P_{00}(t) - \mu_1 P_{10}(t) + \mu_2 P_{11}(t), \\
P'_{11}(t) &= \lambda P_{01}(t) - (\mu_1 + \mu_2) P_{11}(t).
\end{aligned} \tag{5}
$$

In the steady solution, i.e., as $t \to \infty$, $P'_{ij}(t) \to 0$, $P_{ij}(t) \to P_{ij} =$ = const", the differential equations are transformed into algebraic equations:

$$
\begin{aligned}
P_{00}\lambda &= \mu_2 P_{01}, \\
P_{01}(\lambda + \mu_2) &= \mu_1 P_{11} + \mu_1 P_{10}, \\
P_{10}\mu_1 &= \lambda P_{00} + P_{11}\mu_2, \\
(\mu_1 + \mu_2) P_{11} &= \lambda P_{01}.
\end{aligned} \tag{6}
$$

Solving Eqs. (6), we can determine the probabilities for the various states of the control system:

$$
\begin{aligned}
P_{00} &= \frac{\mu_1\mu_2}{(\mu_2+\lambda)(\mu_1+\lambda)}, \\
P_{10} &= \frac{\lambda\mu_2(\mu_1+\mu_2+\lambda)}{(\mu_1+\mu_2)(\mu_2+\lambda)(\mu_1+\lambda)}, \\
P_{01} &= \frac{\lambda\mu_1}{(\mu_2+\lambda)(\mu_1+\lambda)}, \\
P_{11} &= \frac{\mu_1\lambda_2}{(\mu_1+\lambda)(\mu_2+\lambda)(\mu_1+\mu_2)},
\end{aligned}
$$

where λ is the rate of the appearance of new targets in the effective zone of the subject system.

The probability that the target will remain undetected and not fired upon and will carry out its combat assignment is equal to

$$
P_{\text{отк}} = 1 - \frac{\mu_2(P_{01}+P_{11})}{\lambda} = 1 - \frac{\mu_2\mu_1(\lambda+\mu_1+\mu_2)}{(\lambda+\mu_1)(\lambda+\mu_2)(\mu_1+\mu_2)}.
$$

EXAMPLE. On the average, let two targets per unit time (λ = = 2 targets/unit time) appear in the effective band of the reconnaissance system and of the fire control system for "ground-to-ground" units. The reconnaissance system exhibits technical facilities which make it possible in the given area, with the developing combat situation, on the average to detect two targets per unit time (μ_1 = 2 targets/unit time). The control system can process and plan the firing of the facilities at four targets, on the average, per unit time (μ_2 = 4 targets/unit time).

Determine the effectiveness of the system — the probability of firing being conducted at each target at it appears

$$
P = 1 - P_{\text{отк}} = \frac{\mu_2\mu_1(\mu_1+\mu_2+\lambda)}{(\mu_1+\lambda)(\mu_2+\lambda)(\mu_1+\mu_2)} = 0{,}44.
$$

Let us examine the less perfect control system μ_2 = 2 targets/unit time. Then

$$P \cong 0{,}38.$$

The calculations that have been carried out, with consideration of the economy indices, make it possible to select the optimum parameters for the control and reconnaissance systems and to impose reasonable tactical-technical requirements on these.

§4.6. EFFECTIVENESS OF FIRING AT TARGETS AS THEY APPEAR

We will examine the utilization of the apparatus of the theory of queueing with respect to systems exhibiting limited expectation time on the example of a problem involving the determination of the effectiveness of weapons by means of which battle is joined with appearing targets. These targets are understood to refer to fire positions and similar enemy objectives which, on detection, remain in their positions for a limited time. To detect targets the opposing sides make use of a developed reconnaissance system. However, this system is incapable of providing information about all objectives which represent important targets for the attacking side. If we also take into consideration that these objectives maneuver in their positions, and make attempt at camouflage, the fact of the random nature of target detection becomes understandable [130].

In the examination during the course of enemy target detection process by the reconnaissance system of the attacking force (for simplicity we will refer to the other side as the defenders) we can note the absence of a relationship between the fact of the detection of a given object and the number and type of targets which had been detected earlier. We can also agree with the contention that at a given instant of time only a single important target will be detected, rather than several targets, and that during a specific time interval, given the work imposed, in first approximation, the reconnaissance system exhibits a certain average "productivity," i.e., a capacity to detect a specific number of targets per unit time.

On the basis of these assumptions it may be held that the detected targets form a flow which exhibits the properties of the simplest flow with a certain parameter λ. The magnitude of the parameter λ will apparently be a function of the technical equipment of the reconnaissance facilities of the attacking force, the number of enemy objectives in the operating zone of the reconnaissance facilities, the extent of their camouflage, etc.

The time that targets spend in their areas is limited, but not to such an extent that it need not be taken into consideration. The stay times are random quantities. To derive the calculational formulas the target stay time distribution function for the detection zone is assumed to be exponential with the parameter ν, i.e.,

$$h(t) = \nu e^{-\nu t} \ (t > 0),$$

where $\nu = 1/\bar{t}_{ozh}$;

$\bar{t}_{ozh}$ is the average time that a target spends in position after detection.

Under actual combat conditions the distribution function for the target stay time in position after detection may differ from the exponential. However, experience in carrying out a large number of calculations by the method of statistical tests has demonstrated that the basic characteristics of operation for a servicing system (the probability of passing a requisition unserviced, the mathematical expectation of the number of serviced targets) for a steady solution are virtually identical for various distribution functions of the stay time of a requisition (requirement) in the servicing zone.

The time required to fire at each target is also a random quantity. Let us assume that it is distributed exponentially with the parameter μ, i.e.,

$$f(t) = \mu e^{-\mu t} \, (t > 0),$$

where $\mu = 1/\bar{t}_{obs}$,

$\bar{t}_{obs}$ is the average time required to fire at a target.

Losses of weapon facilities by the attacking side due to the defenders' fire are not taken into consideration. Methods to account for answering fire are presented in Chapter 7.

As soon as a target is (targets are) detected, fire is directed at that target immediately by the attacking side. After damage of the target, fire is immediately transferred to other targets, if such are available. If the attacking side is unable to fire at newly appeared targets because it is engaged in firing at targets that have appeared earlier, these targets can remain in the position at which they were detected for a limited time, after which they disappear. Thus, the armament of the attacking side with its reconnaissance facilities and the armament of the defending side represent a queueing system with limited expectation time.

The system considered above may be found in the following states:

A_0 denotes that the attacking side is not firing;

A_1 denotes that one of the weapons is firing, while the remaining weapons are not;

A_k denotes that k weapons are firing;

A_n denotes that all n weapons are firing at their targets;

A_{n+s} denotes that all n weapons are firing, but that s detected targets are not being fired upon.

The number s may be very large and depends on the number of objectives of the defending side that may be located in the sphere of the attacking side's fire effect. We will not dwell on the derivation of the equations of state and their solutions. The reader

can find this in [17]. We will write the formulas for the probabilities of the system states derived for steady conditions:

For state A_0

$$P_0 = \frac{1}{\sum_{k=0}^{n} \frac{\alpha^k}{k!} + \frac{\alpha^n}{n!} \sum_{s=1}^{\infty} \frac{\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}} \quad (0 \leqslant k \leqslant n);$$

for state A_k

$$= \frac{\frac{\alpha^k}{k!}}{\sum_{k=0}^{n} \frac{\alpha^k}{k!} + \frac{\alpha^n}{n!} \sum_{s=1}^{\infty} \frac{\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}} \quad (0 \leqslant k \leqslant n);$$

for state A_{n+s}

$$P_{n+s} = \frac{\frac{\alpha^n}{n!} \frac{\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}}{\sum_{k=0}^{n} \frac{\alpha^k}{k!} + \frac{\alpha^n}{n!} \sum_{s=1}^{\infty} \frac{\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}} \quad (s \geqslant 1),$$

where $\alpha = \frac{\lambda}{\mu}$; $\beta = \frac{\nu}{\mu}$.

A particularly important index of the weapon effectiveness of the attacking side is the probability that each objective of the enemy, detected by his reconnaissance system, is fired upon:

$$P_{обс} = 1 - \frac{\beta}{\alpha} \frac{\frac{\alpha^n}{n!} \sum_{s=1}^{\infty} \frac{s\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}}{\sum_{k=0}^{n} \frac{\alpha^k}{k!} + \frac{\alpha^n}{n!} \sum_{s=1}^{\infty} \frac{\alpha^s}{\prod_{m=1}^{s} (n+m\beta)}}.$$

Hence the probability of destroying each detected objective is equal to

$$W = P_{обс} \cdot P,$$

where P is the probability of damaging the objective when it is fired upon.

To determine $P_{obs} = 1 - P_{otk}$, where P_{otk} is the probability that the target will not be fired upon, we can use the table (see

Table 11 of the appendix) which was prepared by O.A. Novikov. To illustrate the utilization of the derived relationships, let us consider an example.

EXAMPLE. The attacking side has two weapons. To fire at an objective an average of 2 min is required. The probability of damaging the objective with fire is equal to $p = 0.8$. The attacking side has at its disposal a reconnaissance system making possible, on the average, the detection of one target per minute (λ = 1 target/min). The average time that the target spends in position after detection is equal to 4 min (t_{ozh} = 4 min). Determine the effectiveness of the weapon used by the attacking side.

To determine P_{obs} we calculate the parameters:

$$\mu = \frac{1}{t_{обс}} = 0,5\,\frac{\text{targets}}{\text{min}}, \quad \nu = \frac{1}{t_{ож}} = 0,25\,\frac{\text{targets}}{\text{min}},$$

$$\alpha = \frac{\lambda}{\mu} = 2, \quad \beta = \frac{\nu}{\mu} = 0,5.$$

With Table 11 (see the appendix) when n = 2, α = 2, β = 0.5, we obtain

$$P_{отк} = 0,225.$$

The effectiveness of the armament of the attacking side is equal to

$$W = P(1 - P_{отк}) = 0,8\,(1 - 0,225) = 0,62.$$

Varying the fundamental characteristics of the armament (rate of fire, probability of target damage), we can evaluate the effect of these parameters on the effectiveness of the armament used by the attacking side. The method considered above may be employed in evaluating the effectiveness of antiaircraft defense and similar problems.

§4.7. FEATURES IN THE EVALUATION OF THE EFFECTIVENESS OF ANTIAIRCRAFT ARMAMENT AGAINST THE ATTACK OF GROUPED TARGETS

A. The Case in Which the Stay Time for the Target in the Firing Zone is Small

In the previous sections we considered the utilization of the mathematical apparatus of the theory of queueing, developed with respect to unique requirement flows. However, in actual practice there can be cases in which requirements for servicing enter the system in rigorously defined groups — pairs, triplets, etc. On arriving in the queueing system, each of the requirement groups is either serviced or rejected, depending on the extent to which the operator is occupied. As an example of this we can cite the arrival of pairs, flights, etc., of enemy aircraft in an antiaircraft defense system [133]. Each of the enemy aircraft in the area of the antiaircraft defense system protecting an objective will be fired upon by the antiaircraft units. We are confronted with the problem of how to evaluate, under these conditions, the effectiveness of the antiaircraft defense system for the objective

with respect to repelling the aerial attack.

In solving this problem we make the following assumptions:

1. The flow of aircraft groups (pairs, flights) is a Poisson flow. There are m aircraft in each group.

2. The time for firing at each aircraft by the antiaircraft unit is random and subject to an exponential function.

3. Selection of target by each unit is accomplished randomly.

4. The antiaircraft defense system for the objective consists of n antiaircraft units. As soon as the targets appear in the antiaircraft defense zone, they are immediately fired upon. If the antiaircraft units are already engaged in firing, the newly appeared target will penetrate the antiaircraft defense zone unharmed, because the target stay time in this zone is small and commensurate with the time required for it to be fired upon.

5. Consideration of the enemy counteraction is omitted. The antiaircraft defense system may be in various states which we denote:

A_0, which denotes that all antiaircraft units are free;

A_k, which denotes that k units are engaged in firing;

A_n, which denotes that all units are firing.

The probabilities of these states are, respectively, denoted $P_0(t)$, $P_k(t)$, $P_n(t)$. The flow of incoming aircraft groups is assumed to be a Poisson flow and the time required to fire at each target is distributed exponentially; the subject process is therefore a Markov process.

To determine the probabilities of the states $P_0(t)$, $P_k(t)$ and $P_n(t)$, let us derive a system of differential equations. We will derive the equations by the same method as in §4.2; we will therefore not dwell on the intermediate calculations.

State A_0 may occur in the following nonsimultaneous cases:

1. At the instant t all units are free. During the time Δt not a single group of aircraft arrived.

2. At the instant of time t one of the units was engaged in firing. During time Δt this unit concluded firing at the target, but there were no new groups entering the antiaircraft defense zone. Then

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t),$$

where λ is the density of the groups (pairs, flights) in the attack on the objective defended by the antiaircraft system

$$\mu = \frac{1}{t_{\text{обс}}},$$

$\bar{t}_{obs}$ is the average time required by the antiaircraft unit to fire at a single target.

State A_k may occur in the following nonsimultaneous cases:

1. The antiaircraft defense system at time t was in state A_k. During time Δt not one of the units stopped firing and no new enemy aircraft groups appeared in the antiaircraft defense zone.

2. At time t the antiaircraft defense system was in state A_{k+1}. During time Δt no new targets appeared, but one of the units concluded firing. The equation of state is then written as follows:

$$P'_k(t) = -[\lambda + k\mu] P_k(t) + P_{k+1}(t)(k+1)\mu \text{ when } (k+1) \leq n.$$

Let us consider state A_k when $k \geq m$. In this case, in comparison with the previous case, we add yet another possible variant of the state of the system.

At the instant of time t the antiaircraft defense system was in state A_{k-m}. During time Δt a new group of aircraft appeared in the antiaircraft defense zone. None of the units had concluded firing. The equation of state is then written as follows:

$$P'_k(t) = -(\lambda + k\mu) P_k(t) + P_{k+1}(t)(k+1)\mu + \\ + P_{k-m}(t)\lambda \text{ when } k \geqslant m.$$

Finally, it is advisable to consider state A_n, i.e., the state in which all units are engaged in firing, which may occur through several nonsimultaneous ways:

1. At time t the system is in state A_n. During the time interval Δt none of the units disengaged and no new targets appeared.

2. At time t the system was in state A_{n-m}. During time Δt none of the channels became free and one group of targets arrived. The equation of the last state is

$$P'_n(t) = -\mu n P_n(t) + P_{n-m}(t)\lambda.$$

In conclusion, we derive the following system of differential equations:

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t),$$
$$\cdots\cdots\cdots\cdots$$
$$P'_k(t) = -[\lambda + \mu k] P_k(t) + P_{k+1}(t)(k+1)\mu$$
$$\text{when } k < m,$$
$$\begin{aligned} P'_k(t) = &-[\lambda + \mu k] P_k(t) + P_{k+1}(t)(k+1)\mu + \\ &+ P_{k-m}(t)\lambda \text{ when } k \geqslant m, \end{aligned} \tag{1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P'_n(t) = -\mu n P_n(t) + P_{n-m}(t)\lambda.$$

Let us consider the steady case in which $t \to \infty$. In this case $P'_k(t) \to 0$, $P_k(t) \to P_k$ and the solution of the problem reduces to a solution of the system of algebraic equations

$$\begin{aligned} &\alpha P_0 = P_1, \\ &(1+\alpha) P_1 = 2P_2, \\ &\cdots\cdots\cdots \\ &(k+\alpha) P_k = (k+1) P_{k+1} \text{ when } k < m, \\ &\cdots\cdots\cdots\cdots \\ &(k+\alpha) P_k = (k+1) P_{k+1} + \alpha P_{k-m} \text{ when } k \geqslant m, \\ &\cdots\cdots\cdots\cdots\cdots \\ &(n+\alpha) P_n = \alpha P_{n-m}, \text{ when } \alpha = \frac{\lambda}{\mu}. \end{aligned} \tag{2}$$

To this system we add one more obvious condition $\sum_{k=0}^{n} P_k = 1$. The last expression can be presented in the form

$$\sum_{k=0}^{n} P_k = P_0 \sum_{k=0}^{n} f_k(\alpha) = 1.$$

Hence

$$P_0 = \frac{1}{\sum_{k=0}^{n} f_k(\alpha)}.$$

The quantity $\sum_{k=0}^{n} f_k(\alpha)$ can be derived from the recurrence formulas of the system of algebraic equations (2).

EXAMPLE. Let us examine the effectiveness of an antiaircraft system defending a certain typical objective, and let this defense be accomplished by four similar antiaircraft units (n = 4). The objective is being attacked by enemy aircraft. For purposes of comparison let us evaluate the effectiveness of the antiaircraft system when there may be, in each group, m = 1, 2, 3 and 4 aircraft, with a constant attack density of λ = 4 aircraft/min. The antiaircraft units exhibit specific combat characteristics permitting the aerial targets at a specific altitude, with consideration of reloading, to be fired upon at an average rate of fire of $\mu = 1/t_{obs}$ = 2 aircraft/min. The probability of downing a target with the fire from one of the units is equal to P = 0.8. The number of targets participating in the raid is equal to N = 20. Let us determine the parameter $\alpha = \lambda t_{obs}$ = 2. We derive the recurrence formulas for m = 2:

$$P_1 = 2P_0.$$
$$P_2 = 1/2\,(1+\alpha)\,P_1.$$
$$P_3 = 1/3\,(1+2\alpha)\,P_2 - 2P_0.$$
$$P_4 = 1/4\,(1+3\alpha)\,P_3 - 2P_1.$$

After substitution of the values of α we obtain

$$P_1 = 2P_0,$$
$$P_2 = 3P_0,$$
$$P_3 = 10/3P_0,$$
$$P_4 = 19/6P_0.$$

Let us determine the magnitude of P_0:

$$\sum_{k=0}^{4} P_k = 75/6P_0 = 1, \quad P_0 = 2/25.$$

The probability of all units being engaged, i.e., the unharmed passage of the target is then equal to

$$P_{\text{отк}} = 1 - \frac{\mu \sum_{k=0}^{n} kP_k}{\lambda n} = 1 - \frac{2 \cdot 2.45}{2 \cdot 4} = 0.39.$$

B. The Case in Which the Target Stay Time in the Firing Zone is Great

Unlike the previous case, in the problem under consideration we must take into consideration the target stay time in the damage zone of the antiaircraft unit. We will assume that the target stay time in the antiaircraft defense zone is a random quantity subject to an exponential distribution function having the parameter ν. In selecting the distribution function for the target stay time in the antiaircraft defense zone we can be guided by the considerations expressed in §4.6.

We will also assume that the aerial targets group entering the antiaircraft defense zone forms a Poisson flow with the parameter λ. The time required to fire at a target by the antiaircraft unit is a random quantity and is subject to the exponential function with the parameter μ.

As before, we denote the number of units in the antiaircraft defense zone by n and the number of targets in each group by m; the states of the antiaircraft defense system for the objective are as follows:

A_0 denotes that all units are free;

A_k denotes that k units are firing;

A_n denotes that all units are engaged in firing at targets;

A_{n+s} denotes that all units are firing and s new targets have entered the antiaircraft defense zone, but they are not fired upon because the units are engaged.

The probabilities of each of these states are, respectively, denoted by $P_0(t)$, $P_k(t)$, $P_n(t)$, $P_{n+s}(t)$. Let us determine the probability of state A_0. It is possible in the following nonsimultaneous cases:

1. At instant t all units are free. During time Δt not a single enemy group of aircraft appeared.

2. At instant t one of the units was firing, and it concluded firing during Δt. Since the events are nonsimultaneous [incompatible], when $\Delta t \to 0$ we obtain

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t). \tag{3}$$

Let us examine the state $A_k (0 \leqslant k \leqslant m)$. It is possible in the following nonsimultaneous cases:

1. At the instant of time t the antiaircraft defense system was in state A_k. During the time Δt no new targets appeared in the antiaircraft defense zone and none of the units disengaged.

2. At the instant of time t the system was in state A_{k+1}. During the time Δt no new targets appeared in the antiaircraft defense zone, but one of the units finished firing. The differential equation of state is written as follows:

$$P'_k(t) = -(\lambda + k\mu) P_k(t) + (k+1)\mu P_{k+1}(t) \quad \text{when } 0 \leqslant k \leqslant m. \tag{4}$$

Let us examine state A_k for $n \geq k \geq 0$. It will differ from the previous state in terms of what may occur in the third nonsimultaneous case, and namely: at the instant t the antiaircraft defense system was in state A_{k-m}. During the time Δt one more group of targets appeared in the antiaircraft defense zone, but none of the units became free. The differential equation of state $A_k (n \geqslant k \geqslant m)$ is written as follows:

$$P'_k(t) = (\lambda + k\mu) P_k(t) + P_{k+1}(t)(k+1)\mu + P_{k-m}(t)\nu. \tag{5}$$

We have n units at our disposal. It is therefore expedient to regard the state A_n when $(n \geq m)$. This state is possible under the following conditions:

at the instant t the antiaircraft defense system was in state A_n. During the subsequent time interval Δt not a single one of the units became free and no new targets appeared;

at the instant t the antiaircraft defense system was in state A_{n-m}. During the time Δt a group of targets entered the zone, but none of the units became free;

at the instant of time t the antiaircraft defense system was in state A_{n+1}. During the time Δt no new targets appeared,

but either one of the units became free, or one of the aircraft departed unharmed from the zone. The equation of state is then

$$P'_n(t) = -(\lambda + n\mu)P_n(t) + P_{n-m}(t)\lambda + \\ + P_{n+1}(t)(n\mu + \nu). \tag{6}$$

Let us examine state A_{n+s} in detail, this state being possible in the following cases:

at the instant t the antiaircraft defense system was in state A_{n+s}. No new targets appeared in the zone during time Δt, none of the targets escaped without being fired upon, and none of the units became free;

at the instant t the antiaircraft defense system was in state A_{n+s-m}. A new group of targets appeared in the zone during the time Δt, not one of the enemy aircraft escaped the zone without having been fired upon and none of the units disengaged from firing;

at the instant of time t the antiaircraft defense system was in state A_{n+s+1}. No new targets appeared during the time Δt, but either one aircraft escaped the zone unharmed, or one of the units disengaged from firing. The equation of this state is as follows:

$$P_{n+s}(t) = \lambda P_{n+s-m}(t) - P_{n+s}(t)(\lambda + n\mu + s\nu) + \\ + P_{n+s+1}(t)[n\mu + (s+1)\nu]. \tag{7}$$

Let us consider the steady solution for which we assume that when

$$t \to \infty \quad P'_k(t) \to 0, \text{ and } P_k(t) \to P_k = \text{const},$$

where $k = 1, 2, \ldots, n, \ldots, (n + s)$.

In this case we obtain a system of algebraic equations

$$\begin{aligned} \lambda P_0 &= \mu P_1, \\ (\lambda + \mu)P_1 &= 2P_2\mu, \\ &\ldots\ldots\ldots\ldots \\ (\lambda + m\mu)P_m &= (m+1)\mu P_{m+1} + P_0\lambda \text{ when } k = m, \\ &\ldots\ldots\ldots\ldots \\ (\lambda + k\mu)P_k &= (k+1)\mu P_{k+1} + P_{k-m}\lambda \text{ when } k \geqslant m, \\ &\ldots\ldots\ldots\ldots \\ (\lambda + n\mu)P_n &= (n\mu + \nu)P_{n+1} + P_{n-m}\lambda, \\ &\ldots\ldots\ldots\ldots \\ (\lambda + n\mu + s\nu)P_{n+s} &= [n\mu + (s+1)\nu]P_{n+s+1} + \\ &+ \lambda P_{n+s-m} \text{ when } 1 \leqslant s \leqslant \infty. \end{aligned} \tag{8}$$

We denote $\lambda/\mu = \alpha$ and $\nu/\mu = \beta$. Equation (8) then is simplified somewhat

$$\begin{aligned}
&\alpha P_0 = P_1, \\
&(\alpha+1) P_1 = 2P_2, \\
&\dots\dots\dots\dots \\
&(\alpha+m) P_m = (m+1) P_{m+1} + \alpha P_0 \text{ when } k = m, \\
&\dots\dots\dots\dots \\
&(\alpha+k) P_k = (k+1) P_{k+1} + \alpha P_{k-m} \text{ when } k \geqslant m, \\
&\dots\dots\dots\dots \\
&(\alpha+n) P_n = (n+\beta) P_{n+1} + \alpha P_{n-m} \text{ when } n > m, \\
&\dots\dots\dots\dots \\
&(\alpha+n+s\beta) P_{n+s} = [n+(s+1)\beta] P_{n+s+1} + \\
&\qquad + \alpha P_{n+s-m} \text{ when } 1 \leqslant s \leqslant \infty.
\end{aligned} \tag{9}$$

Hence

$$P_k = P_0 f_k(\alpha, \beta, m),$$

where $f_1(\alpha, \beta, m) = \alpha$; $f_2(\alpha, \beta, m) = \frac{1}{2}\alpha(\alpha+1)$ etc.

The sum of the probabilities for all possible states of the system is equal to

$$\sum_{k=0}^{\infty} P_k = 1. \tag{10}$$

This sum can be presented as follows:

$$\sum_{k=0}^{\infty} P_k = P_0 \sum_{k=0}^{\infty} f_k(\alpha, \beta, m). \tag{11}$$

Hence

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} f_k(\alpha, \beta, m)}. \tag{12}$$

The quantity $\sum_{k=0}^{\infty} f_k(\alpha, \beta, m)$ can be derived from the recurrence formulas of the system of equations (9).

Let us determine the average number of targets situated in the antiaircraft defense zone that have not been fired upon

$$\overline{S}(\alpha, \beta, m) = \sum_{s=1}^{\infty} SP_{n+1}. \tag{13}$$

The probability of an enemy aircraft passing through the antiaircraft defense zone without having been fired upon is equal to

$$P_{\text{отк}} = 1 - \frac{\mu \sum_{k=0}^{\infty} kP_k}{\lambda m}. \tag{14}$$

EXAMPLE. The antiaircraft defense system of the objective consists of 5 antiaircraft units. Each of these spends, on the average, t_{obs} = 1 min on firing at an aerial target. The average

stay time in the antiaircraft defense zone is equal to t_{ozh} = 1 min. The enemy aircraft are attacking the objective in pairs (m = = 2). The attacking rate of paired targets is equal to λ = 2 target pairs/min. Evaluate the effectiveness of the antiaircraft defense system if we assume that each of the units can fire simultaneously only at a single one of the enemy aircraft. The number of targets in the raid is 20.

Solution. Let us determine the auxiliary parameters:

$$\mu = 1/t_{\text{обс}} = 1\,\frac{\text{aircraft}}{\text{min}},$$

$$\nu = 1/t_{\text{ож}} = 1\,\frac{\text{aircraft}}{\text{min}},$$

$$\alpha = \frac{\lambda}{\mu} = 2,$$

$$\beta = \frac{\nu}{\mu} = 1.$$

We derive the system of algebraic equations

$$\begin{aligned}
2P_0 &= P_1,\\
(2+1)\,P_1 &= 2P_2,\\
(2+2)\,P_2 &= 3P_3 + 2P_0,\\
(2+3)\,P_3 &= 4P_4 + 2P_1,\\
(2+4)\,P_4 &= 5P_5 + 2P_2,\\
(2+5)\,P_5 &= 6P_6 + 2P_3\\
(2+5+1)\,P_6 &= (5+2\cdot 1)\,P_7 + 2P_3,\\
(2+5+2)\,P_7 &= (5+3\cdot 1)\,P_8 + 2P_4,\\
(2+5+3)\,P_8 &= (5+4\cdot 1)\,P_9 + 2P_5,\\
(2+5+4)\,P_9 &= (5+5\cdot 1)\,P_{10} + 2P_6,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Neglecting the small values of P_{12} and the probabilities of higher sequential numbers, we obtain

$$1 \approx \sum_{k=1}^{20} P_k = 19P_0, \quad P_0 = \frac{1}{19}.$$

The probability of an enemy aircraft passing through the antiaircraft defense zone without having been fired upon is equal to

$$P_{\text{отк}} = 1 - \frac{\mu \sum\limits_{k=0}^{5} kP_k}{\lambda m} =$$

$$= 1 - \frac{1\,(1\cdot 0{,}105 + 2\cdot 0{,}153 + 3\cdot 0{,}175 + 4\cdot 0{,}165 + 5\cdot 0{,}135)}{2\cdot 2} \approx 0{,}46.$$

§4.8. DETERMINATION OF THE TRANSMISSION CAPACITY AND LOADS OF REPAIR WORKSHOPS

Repair workshops frequently service units and subunits situated at considerable distances from each other. To provide immediate repair of malfunctioning equipment, it would be possible

to have repair workshops attached to each subunit. However, this is not always feasible, since the personnel of such workshops may exceed the repair requirements of such a subunit. It is therefore advantageous for the repair of specific forms of combat equipment, consisting of the armament of many subunits, to employ a number of specialized workshops. The number of such workshops is determined from the calculation that they should not remain idle and without work, while at the same time not forming a bottleneck in the flow of requisitions for repair.

It is extremely difficult to solve such a problem by qualitative considerations without a quantitative evaluation. In peacetime the required number of such workshops can be determined by a sampling method. For this we can attach a specific number of repair workshops to several subunits. Experience in their operation over a specific period of time will show the validity of the adopted decision. However, with the passage of time the equipment will change, particularly military equipment, and new and more advanced forms of armament will appear. At the same time the repair of equipment does not remain at the same level; experimental determination of the required number of repair workshops and their personnel may therefore require such a long period of time that it will have an unfavorable effect on the utilization of the armament by the troops. There is no doubt that such an experiment would be intolerable during combat operations.

To solve this problem we have to use quantitative methods of analysis by means of which it can be properly solved for a significantly smaller required number of statistical data. In this case it is advisable to use the apparatus of the theory of queueing, developed in connection with systems with expectation for a limited number of servicing units [66]. As was mentioned earlier, before proceeding with the solution of the stated problem, we must have a specific quantity of statistical data. Among these data we should include:

the average time required to detect faults;

the average time required to carry out basic operations in the technological process of armament repair;

the time required for the requisition and arrival of a mobile repair workshop at the subunit (for semistationary workshops – the time required to deliver the armament to the repair station), etc.;

the frequency of armament breakdown.

After the statistical processing of the derived data we can obtain the basic parameters characterizing the armament repair system.

These include:

the density of the armament breakdown flow (the repair requisition flow) λ;

the average time required to requisition the workshop and to carry out the repair.

Determination of the assumed equipment breakdown flow under combat conditions must be accomplished by means of various forecasting methods, with consideration given to the nature of the combat operations; we will not dwell on this point at this time. With the required statistical data available, we can proceed to the solution of the problem.

A. Statement of the Problem

Let there be n repair workshops for a specific form of armament, which are distributed among various units and subunits. Combat equipment that has broken down is repaired by one of the repair workshops. The servicing may be organized for accomplishment by the personnel of mobile workshops which may be directed at any time to the subunit where repair requirements exist.

The repair operation may be organized at large workshops with a well organized technological flow to which malfunctioning equipment will be sent.

In either case, the time required for the repair of the malfunctioning armament will be composed differently. In the first case it will consist of the time required to call for the workshop, its movement to the repair position and the time required to set up the workshop there, and the time required to carry out determination of the malfunction and the actual repair. In the second case, it will be determined by the time required to deliver the malfunctioning armament to the rear repair workshop, to inspect that armament for defects and to carry out the repairs. In the subject cases the corresponding servicing time components will vary. Let us assume that the servicing time is a random quantity with an exponential distribution function having the parameter ν, where

$$\nu=\frac{1}{\bar{t}_р},$$

$$\bar{t}_р=\bar{t}_{выз}+\bar{t}_{деф}+\bar{t}_{дв\ м}+\bar{t}_{разв}+\bar{t}_{рем}+\bar{t}_{св};$$

$\bar{t}_{vyz}$ – the average time to call for the workshop;

$\bar{t}_{def}$ – the average time to determine the malfunction;

$\bar{t}_{dvm}$ – the average time for the movement of the workshop to the repair location;

$\bar{t}_{razv}$, $\bar{t}_{sv}$ – the average time to set up and dismantle the workshop;

$\bar{t}_{rem}$ – the average time for repair;

$\bar{t}_r$ – the average time to repair the malfunctioning armament.

We determine $\bar{t}_r$ for semistationary repair workshops in an analogous manner.

The flow of incoming repair requisitions is limited by the number of subunits being serviced for repair of combat equipment, and this flow is assumed to be the simplest. This assumption is valid for the following reasons:

1. The instants of armament breakdown and entry into the repair sphere are independent events occurring at nonintersecting time intervals.

2. The breakdown of a given weapon is independent of the number of such weapons that broke down earlier.

3. The number of incoming repair requisitions depends on the density, i.e., on the average anticipated number of requisitions λ per unit time. We assume that if a requisition has come to the repair sphere, the workshop is immediately assigned to the corresponding subunit. If all workshops are engaged, the weapon that has broken down will wait its turn for the completion of the repair.

B. Basic Indicators for a Repair Servicing System

For a quantitative evaluation of an armament repair servicing system we can use the following relationships [66]:

1. The probability that all workshops are not engaged in repair

$$P_0=\frac{1}{\sum\limits_{k=0}^{n}\frac{m!\alpha^k}{k!(m-k)!}+\sum\limits_{k=n+1}^{m}\frac{m!\alpha^k}{n^{k-n}n!(m-k)!}}, \qquad (1)$$

where $\alpha = \lambda/\nu$;

λ is the parameter for the flow of incoming repair requisitions;

$\nu = 1/t_r$;

$\bar{t}_r$ is the average time required to repair the malfunctioning equipment;

m is the number of weapons in the units and subunits being serviced with repair;

n is the number of repair workshops (technological repair flows).

2. The probability that k workshops are engaged in the repair of combat equipment:

$$P_k=\frac{m!\alpha^k P_0}{n!(m-k)!} \quad \text{when } 1\leqslant k\leqslant n. \qquad (2)$$

3. The same probability for the condition $k > n$

$$P_k=\frac{m!\alpha^k P_0}{n^{k-n}n!(m-k)!}. \qquad (3)$$

4. The probability that all workshops are engaged in the repair of armament, or the probability of rejection of immediate

repair

$$P_n = \frac{m!\,\alpha^n P_0}{n!\,(m-n)!}. \qquad (4)$$

5. The average number of armament items being repaired and awaiting repair:

$$M_1 = \left[\sum_{k=1}^{n} \frac{m!\,\alpha^k}{(k-1)!\,(m-k)!} + \sum_{k=n+1}^{m} \frac{km!\,\alpha^k}{n^{k-n}\,n!\,(m-k)!}\right] P_0. \qquad (5)$$

6. The average number of weapons which will await repair (the average waiting turn) because the workshops are occupied:

$$M_2 = \sum_{k=n+1}^{m} \frac{(k-n)\,m!\,\alpha^k}{n^{k-n}n!\,(m-k)!} P_0. \qquad (6)$$

7. The average percentage of armament awaiting repair:

$$K_1 = \frac{M_2}{m}\cdot 100\% = \sum_{k=n+1}^{m} \frac{(k-n)\,(m-1)!\,\alpha^k}{n^{k-n}\,n!\,(m-k)!} P_0 \cdot 100\%. \qquad (7)$$

8. The average number of workshops not engaged in repair

$$M_3 = \sum_{k=0}^{n} \frac{(n-k)\,m!\,\alpha^k}{k!\,(m-k)!} P_0. \qquad (8)$$

9. The percentage of idle workshops

$$K_2 = \frac{M_3}{n}\,100\% = \left[\sum_{k=0}^{n-1} P_k - \frac{1}{n}\sum_{k=0}^{n-1} kP_k\right] 100\%. \qquad (9)$$

Let us consider the examples. There are three mobile workshops (n = 3) to repair 10 weapons (m = 10). Experience in the operation of these has demonstrated that on the average each of the weapons will break down once a month, i.e., λ = 1 weapons/ /month. To call a workshop and to repair the equipment an average of about six days is required for ν = 5 weapons/month. We are required to determine the basic characteristics for the organization of an armament repair system.

First of all, let us determine the probability that all of the workshops are not engaged in repair. For this we determine the quantity $\alpha = \lambda/\nu$ = 0.2 and compile Table 4.8.1 of the calculations.

It follows from Table 8.4.1 that the probability that all of the workshops will not be engaged in repair is equal to

$$P_0 = 0{,}155.$$

That means that on 4-5 days of the month all of the workshops will be free, and their technical personnel may rest.

TABLE 4.8.1

k	$\frac{P_k}{P_0}$	P_k	kP_k	$(k-n)P_k$	$(n-k)P_k$
0	1	0.1548	0	—	0.4644
1	2	0.3096	0.3096	—	0.6192
2	1.8	0.2786	0.5572	—	0.2786
3	0.96	0.1486	0.4458	—	0
4	0.448	0.0693	0.2772	0.0693	—
5	0.179	0.0277	0.1385	0.0554	—
6	0.056	0.0087	0.0522	0.0261	—
7	0,012	0.0018	0.0126	0,0072	—
8	0,0025	0.0004	0,0032	0,0020	—
9	0.0004	0,0001	0,0009	0,0006	—
10	0	0	0	0	—
	$\Sigma = 6.4579$	$\Sigma = 1.0$	$M_1 = 1.7972$	$M_2 = 0.1606$	$M_3 = 1.3622$

However, this does not indicate that there will no "line" of defective armament requiring repair. The "length" of this line, of course, will vary in various periods. On the average, the number of weapons expecting repair will be equal (see the sum of the fifth column in Table 4.8.1) to

$$M_2 = 0{,}16 \text{ weapons.}$$

Hence the average percentage of armament expecting repair is equal to

$$K_1 = \frac{M_2}{m} 100\% = 1{,}6\%.$$

Let us see the extent to which the workshops are efficiently employed. The average number of workshops not engaged in repair is equal to (see the 6th column in Table 4.8.1)

$$M_3 = 1{,}36 \text{ workshops,}$$

and the idleness factor is equal to

$$K_2 = \frac{M_3}{n} 100\% = \frac{1.36}{3} \cdot 100\% = 46\%,$$

i.e., very high.

Let us determine the average number of weapons which are either in repair or awaiting repair, or more precisely, the average number of weapons not suitable for combat:

$$M_1 = 1{,}79 \text{ weapons.}$$

Hence the average percentage of weapons not suitable for combat is equal to

$$K_3 = \frac{M_1}{m} 100\% = 17{,}9\%.$$

This is a very high percentage of unsuitability for combat and it is determined primarily by the time that the weapon is in repair (the percentage of weapons "expecting" repair is small: k = 1.6%).

Let us consider another example for the same armament operating conditions, but with the repair operation organized differently. Instead of three mobile workshops to service the weapons, one semistationary repair workshop has been set up with three well organized technological flows. However, in this case the average time required to deliver the malfunctioning weapon to the workshop is several times greater than the time required to call for the workshop and for the arrival of the mobile repair workshop in the subunit. And despite the reduction in the duration of the actual repair, the total time that the weapon spends in the repair sphere is increased by a factor of 2.5. Then ν = 2 weapons/month, α = 0.5.

As before, the required calculations are presented in the form of a table.

TABLE 4.8.2

k	$\frac{P_k}{P_0}$	P_k	kP_k	$(k-n)P_k$	$(n-k)P_k$
0	1	0.010	0	—	0.030
1	5	0.051	0.051	—	0.102
2	11.25	0.115	0.230	—	0.115
3	15.0	0.153	0.459	—	0
4	17.5	0.178	0.712	0.178	—
5	17.47	0.178	0.890	0.356	—
6	14.56	0.149	0.794	0.447	—
7	9.7	0.099	0.693	0.396	—
8	4.85	0.050	0.400	0.250	—
9	1.58	0.016	0.144	0.096	—
10	0.28	0.003	0.030	0.021	—
	$\Sigma = 98.19$	$\Sigma = 1.002$	$M_1 = 4.403$	$M_2 = 1.744$	$M_3 = 0.247$

The results of the calculations are expediently summarized in the single Table 4.8.3 for purposes of comparison.

TABLE 4.8.3

Characteristic of variant	P_0	M_2	K_2 %	M_3	K_3 %	M_1	K_1 %
Mobile workshops	0.155	0.16	1.6	1.36	46	1.79	17.9
Semistationary workshops	0.01	1.74	17.4	0.25	25	4.4	44

It follows from the tabular data that in connection with the general reduction in the transmission capacity of the repair facilities there is a pronounced increase in their work load for a constant density of incoming requirements for weapons repair: the probability of idleness without repair for all workshops (technological flows) P_0 diminished by a factor of ~16, the number of workshops not engaged in repair (technological flows) M_3 was reduced by a factor of 5. However, the average number of weapons not suitable for combat increased from K_1 = 17.9% to K_1 = 44%. Under combat application conditions there is a pronounced increase in the rate of weapons breakdown. With respect to the case of providing repair services with mobile workshops we assume that the flow of repair requirements increased by a factor of 5. We

offer a table of comparative results.

TABLE 4.8.4

Value of λ	P_0	M_3	K_3 %	M_2	K_2 %	M_1	K_1 %
$\lambda = 0.2$	0.155	0.16	1.6	1.36	46	1.798	17.98
$\lambda = 1$	0.0005	4.02	40.2	0.012	0.4	7.00	70

As follows from the data of the table, the work load of the workshops increased markedly (we compare M_3, K_3, P_0). In the last case the engineers and technicians will have virtually no free time. However, despite this, the percentage of weapons not fit for combat increased markedly from 18 to 70%. It is obvious that in the last case the available number of workshops is clearly inadequate to provide for armament repair.

This method may be used to evaluate the effects on the transmission capacity of repair facilities produced by perfecting repair technology (reducing $\bar{t}_{def}$, $\bar{t}_{rem}$), by improving workshops mobility $\bar{t}_{dvm}$, by improving the set-up time for repair and dismantling time after repair, and similar factors. On the basis of an economic evaluation we can find the optimum armament repair procedure and we can determine the most advantageous number of various forms of workshops and the suitability of introducing new repair methods.

§4.9. THE SOLUTION OF MILITARY PROBLEMS ASSOCIATED WITH QUEUEING BY THE METHOD OF STATISTICAL TESTS

In the previous sections we presented solutions for certain problems in queueing which were carried through to calculational formulas. This became possible through a number of significant assumptions: the steadiness and uniqueness of the requisition flows, the absence of aftereffects, the simplest distribution functions were taken for the operation of the servicing system elements, etc. These assumptions are not always in agreement with practice. Moreover, the operation of actual queueing systems is accompanied by the breakdown of servicing devices for various reasons, their repair, the presence of rejects in servicing (in military affairs, failure to damage target), etc. All of these difficulties can be overcome by employing the method of statistical tests [12]. In solving the problems of queueing by the method of statistical tests we note several stages.

The assignment of the requirements (requisitions) flow. In the modeling of a requisitions flow we find possible not only random sequences, but determined sequences. The features of the latter are determined by the nature of the enemy's use of combat units. For example, an enemy air force attack under certain conditions can be represented in the form of rather rigorously organized combat formations of air force facilities. The methods of assigning the various random sequences and of their realization in computers are shown in §2.2 of this book, as well as in [12, 13].

The time required for the servicing of each requisition, the time of arrival of each requisition in the servicing zone, etc., are random quantities with their own distribution functions. The methods of assigning random quantities are rather well worked out and discussed in §2.2 of this book.

The most important is the modeling of the actual servicing process which is presented in the form of an algorithm — a collection of mathematical and logical rules and limitations. If the servicing process is not too complex, and the needed number of realizations is not large, it can be calculated by hand on paper. Otherwise, we have to turn to the electronic computers (see §2.2). After the calculations have been carried out and after we have processed many of the realizations of the process, we carry out the statistical processing of the results. As quantities which are exponents of the servicing quality we can take those which have gained greatest acceptance [12, 66].

For systems with failure the average percentage of failures in servicing during a specific time interval (t_0, t)

$$M_{\text{отк}}(t_0, t) = \frac{n(t_0, t)}{N(t_0, t)},$$

where $n(t_0, t)$ is the average number of failures in the realizations during the time t_0, t;

$N(t_0, t)$ is the average number of requirements during this same time segment.

For systems with expectation: $T_{\text{ozh}}(t_0, t)$ is the average time of expectation for a requisition in line during the time interval (t_0, t); $M_{\text{o}}(t_0, t)$ is the average length of the line during this same time segment.

For systems with limited expectation time we can use all of the above described indicators. In solving military problems we have to determine other important indicators as well. These include the consumption of missiles, the consumption of rockets for each downed target, the distribution of downed targets over antiaircraft defense zones, the possibilities of ammunition supplies, etc.

Manuscript Page No.	Transliterated Symbols
210	ож = ozh = ozhidaniye = expectation
215	обс = obs = obsluzhivaniye = servicing
221	проп = prop = propushchennyy = passed [permitted to pass]

221 сб = sb = sbit = downed

222 нал = nal = nalet = attack [raid]

224 н = n = nalet = attack [raid]

234 отк = otk = otkaz = failure

248 выз = vyz = vyzov = call

248 деф = def = defektatsiya = inspection

248 двм = dvm = dvizheniye masterskoy = workshop transfer

248 разв = razv = razvertyvaniye = setting up

248 св = sv = svertyvaniye = packing up

248 рем = rem = remont = repair [overhaul]

248 р = r = remont = repair [overhaul]

254 о = o = ochered' = line